Le Grand

PH2101 compilation note

Credits

Diptanuj, Adrika, Piyush, Rangeet sir

Don't Panic

bye

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Class Sem III Roll 22MS038

Subject PH2101

School/College

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PH2101

Waves and Optics

1st Aug 2023

Syllabus →
- SHM
- Oscillation of multiple DOF } Mod 1.
- Oscillation of string (beaded) (Modes, Fourier Analysis) } Mod 2
- Forced Oscillation. } Mod 3

Mech I dependent.

Keep these in mind must happen simultaneously.

- Travelling waves
- Light as wave.
  - (I) (No geo opt, wave opt and boundary stuff)
  - (II) Polarization, interference, diffraction.

EM reqd. Also Griffiths

Immense mistake of not doing year 1 physics seriously will catch up.

(*) Tutorial class might shift.

No tut this week → REGULAR class

Grading ⇒ Fixed in LVL 2.

Internal → 30% → Breakup at ba. (Might have attendance)

MS → 20%.

ES → 50%.

⊛ Recheck before midsem

⊛ 3 to 4 class tests

Topics to be split into pre-MS and post-MS

→ Till wave eqn, travelling stuff

→ The rest

CT during tut hours.

⊛ Prob sets will be given, solved during Tutorials.

⊛ Office Hours on appointment.

Main Ref: Crawford → Primary → This is what I will be reading.

A.P French.

Discussion regarding why we study this. ⊛ most of this is qualitative discussion - not of much importance.

We have states in physics → this includes bound states.

electrons in atoms have bound states that allow certain energy levels.

How to create monochrom light? Use prism to disperse.

What do we do when we try to measure intensity of light?

when we measure with increasing accuracy, we see that we measure nothing less than,

$h\nu$ → But ν is continuous? → Suggests quantisation of EM waves.

Light → Boson → Jargon

material → Fermion.

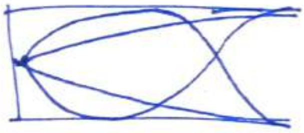
Δ Quantum particle →

Quant particle has bound state, it has discrete energy.

What if it is free? We can accel it to any value, so it is also discrete here?

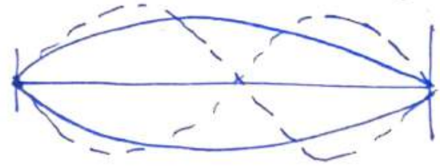
It is also ~~dis~~ discrete. → ? Don't understand, can't state, thus do not care.

Example of waves in flute →



→ Only certain waves are allowed, as node at one end, and antinode at another.

Same with string attached at two ends.



So classical objects are quantised too.

Or, well, at least they exhibit behavior akin to that.

eigenval/func/waves are crucial and repeated constantly.

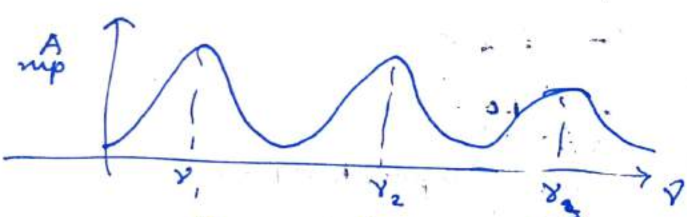
↳ These are somehow important to analysis of waves

This is why the course was created.

Interference, vibration, polarization etc also pop up in all fields of science. → So this is important.

⊛ Physical systems in oscillation provide least resistance / ~~ap~~ amplifies at natural freqs.

The case for air column; ν = Driving freq (sound)



Modes of air colum.

Some systems have several fundamental modes of oscillation

Something like a pendulum has only one normal mode. → Some have just one.

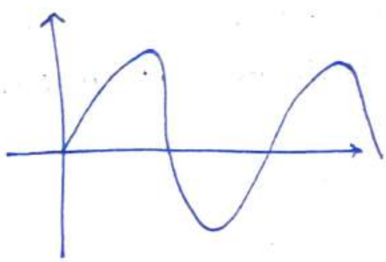
⊛ If we have EM waves, which are free (Not bound), how are they quantised?

Energy of EM wave $\propto E^2 + B^2$.

We quantise fields.
Uses Hamiltonians and other things that I do not know, and thus do not care about atm.

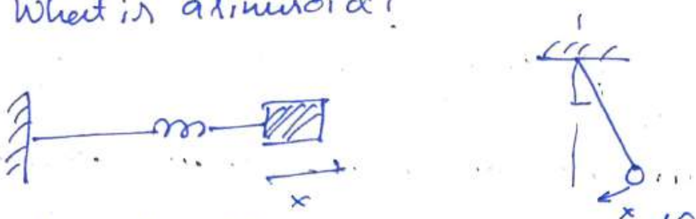
We start with no pre-req. (I mean, yeah) 2nd August 2023

☐ Simple Harmonic Motion →



Can be represented as a sinusoidal wave.

What is a sinusoid?

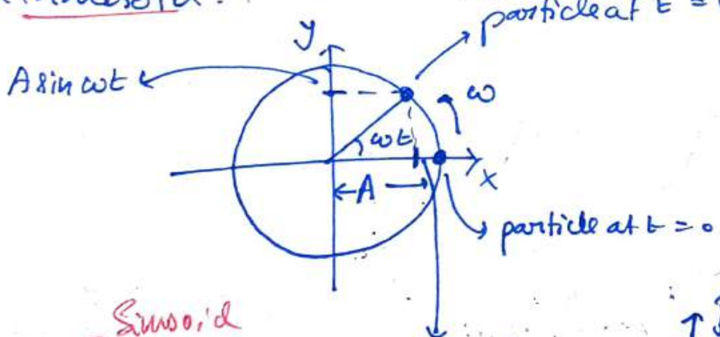


We should be able to represent the displacement as

$$\boxed{x(t) = \underbrace{A}_{\text{Amplitude}} \cos(\underbrace{\omega}_{\text{Ang freq}} t + \underbrace{\phi}_{\text{phase}})}$$

→ Definition of SHM

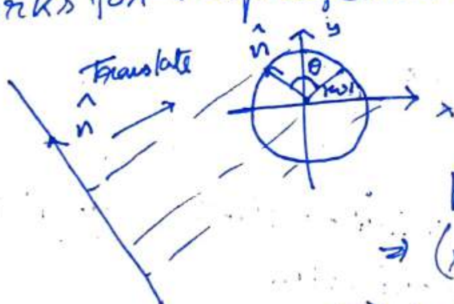
Observation → If you have a point rotating with uniform angular speed, its projection on a line is a sinusoid. ↳ Any line



$\uparrow \hat{j}, \rightarrow \hat{i}$

$$\left.\begin{aligned} x &= A\cos\omega t \\ y &= A\sin\omega t \end{aligned}\right\} \text{Sinusoid} \implies \vec{r} = x\hat{i} + y\hat{j}$$

But this works for the projection on any line.



θ is arbitrary.

$\hat{n} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

projection → $\vec{r}\cdot\hat{n}$

$\Rightarrow (x\hat{i} + y\hat{j})\cdot(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$\Rightarrow \underbrace{x\cos\theta + y\sin\theta}_{\text{Sinusoid}}$

We can simplify and write as,

$$= A\left(\cos\omega t \cdot \cos\theta + \sin\omega t \cdot \sin\theta\right)$$

$$= A\cos(\omega t - \theta)$$ → Sinusoid

(*) We can thus use a circular motion to represent an SHM.

☐ Complex representation of rotation → Is this identification unique?

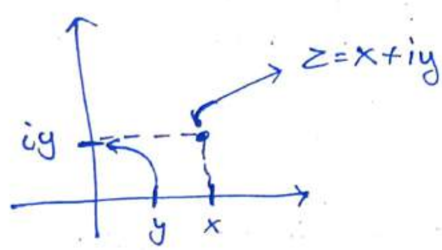
$$\vec{r} = A\cos\omega t\,\hat{i} + A\sin\omega t\,\hat{j}$$
$$= x\hat{i} + y\hat{j}$$

Instead of this projection vector, we choose a representation,

$$\boxed{x + iy \quad , i = \sqrt{-1}}$$ → wow, did not know.

How does this work? By association — treat the Argand plane as the cartesian plane that is. ↗ Easy enough to see.

(*) Multiplying by i in arg plane is rotation counterclockwise by 90°



We can represent the vector now as,

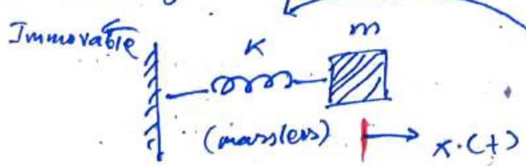
$$r = A\cos\omega t + i\sin\omega t$$
$$\Rightarrow r = A(\cos\omega t + i\sin\omega t)$$
$$\Rightarrow \boxed{r = Ae^{i\omega t}}$$

So, the position vector for the rotation can be represented this way.

We may also write it as,

$$r = Ae^{(i\omega t + \phi)}$$ → phase

We now take a system, spring mass.



Spring const.

We write the equation of motion,

$$m\frac{d^2}{dt^2}x(t) = F = -Kx,$$ → Basic Hooke's Law. (Not always true — only approx)

↓ $m\ddot{x}$ (easier notation)

In real life,

$$F = -Kx + \alpha x^2 + \beta x^3 + \ldots$$

mostly small

$$\Rightarrow m\ddot{x} = -Kx$$

Suppose we have the non-linear terms,

$$m\ddot{x} = -Kx + \alpha x^2 + \beta x^3$$

We have difficulties. Before that, we have the difficulty of the principle of superposition.

* If we have a lin eqn, that has two soln, then the sum of soln is also a solution ↳ (any linear comb) ↳ of the solutions

If $x_1(t)$ and $x_2(t)$ are solutions to the eqn, That is.

$$m\ddot{x}_1 = -Kx_1 + \alpha x_1^2 + \beta x_1^3 + \cdots$$

$$m\ddot{x}_2 = -Kx_2 + \alpha x_2^2 + \beta x_2^3 + \cdots$$

$$\Rightarrow m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1 + x_2) + \alpha(x_1^2 + x_2^2) + \beta(x_1^3 + x_2^3) + \cdots$$

⇒ Linearity breaks down, thus principle of superposition does not hold. (why? why do we need to hold it).
↳ ? □ Look in to it;

Ex of other non-linear motions is giant waves in the ocean (strongly non-linear)

They have weird solutions.

So, we ignore the non-linear terms (till 4th year lol)

$$m\ddot{x} = -Kx$$

Now, $x = Ae^{i\omega t}$

We can plug it into the diff eqn to solve, but the complex quantity we get at the end is complex and the projection needs to be taken. Done by, (either)

① Taking the real part

② Fitting in initial values — They are real and essentially restrict the value to real lin.

$$\therefore -m\omega^2 A e^{i\omega t} = -KAe^{i\omega t}$$

clockwise/anticlockwise.

$$\Rightarrow \omega^2 = \frac{K}{m} \Rightarrow \boxed{\omega = \left\{\pm \sqrt{\frac{K}{M}}\right.}$$

2 solutions.

We take a linear combination of the two solutions,

$$\boxed{x(t) = c_1 e^{i\sqrt{\frac{k}{m}}t} + c_2 e^{-i\sqrt{\frac{k}{m}}t}}$$

Call ω Ang freq of oscillation.

Just convention.

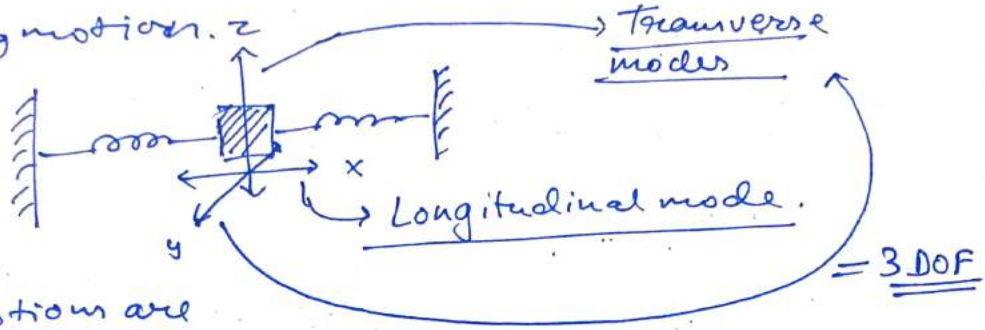
Being more specific,

$$x(t) = Re\left[c_1 e^{i\sqrt{\frac{k}{m}}t} + c_2 e^{-i\sqrt{\frac{k}{m}}t}\right]$$

Or put in initial condition if we had it.

☐ Oscillations with multiple degrees of freedom →

Before, the oscillation of mass could be only along the spring motion.



y and z motions are similar → This is because modes that have same freq solution are called degenerate modes

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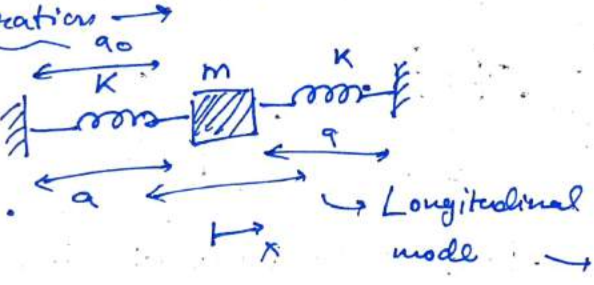
Today we will study modes of vibration.

4th August 2023

☐ Modes of vibration →



We will study motion of the mass.

Springs are identical.

Nothing special in name, but if we have 'connections' we call motion colinear to that as longitudinal, and ⊥ to that as transverse.

Now in equilibrium state.

a = equilibrium distance when not attached.

a_0 = equilibrium distance for spring when not attached

Force due to a spring on mass $= K(a-a_0)$

Equation of motion along x,

$$m\ddot{x} = -K(a-a_0+x) + K(a-a_0-x)$$

Opp direction.

Due to x being +ve, contraction for spring on right.

$\Rightarrow m\ddot{x} = -2Kx$

Force balance exists already in eq. state $\rightarrow$ we care only about the excess force. $\rightarrow$ $m\ddot{x} = -Kx - Kx = -2Kx$
also gives same result

But, it is physically important to know that the cancellation happens. — sometimes it does not (say, in the case where the springs are not identical) $\boxed{}$ Solve non-identical case

Now, since this oscillates,

$x = Ae^{i\omega t}$

Substituting,

$\dot{x} = iA\omega e^{i\omega t}$

$\ddot{x} = -A\omega^2 e^{i\omega t}$

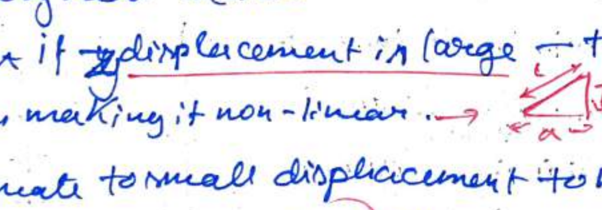
$\therefore -m\omega^2 x = -2Kx$

$\Rightarrow \omega^2 = \frac{2K}{m} \Rightarrow \omega = \pm\sqrt{\frac{2K}{m}} = \pm\,\omega_0$

$\therefore$ General solution,

$x = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}$ $\rightarrow$ Longitudinal motion

Now, transverse,



⊛ This is not very real. We said that we will only do linear motion. But if ~~y~~ displacement is large — the spring length changes making it non-linear. $\rightarrow$

$\Rightarrow L = \sqrt{a^2+y^2}$

So we approximate to small displacement to keep things linear.

$\cos\theta$ com

$$m\ddot{y} = -K(L - a_0)\frac{y}{L} - K(L - a_0)\frac{y}{L}$$

$\Rightarrow m\ddot{y} = -2K(L - a_0)\frac{y}{L}$ $\rightarrow$ Exact, but not linear.

Why? $L = \sqrt{a^2+y^2}$, substituting gives non-linear RHS in ODE

What we do, is $y << a$ (Not extra assumption, as we said it to frame eqn)

$\Rightarrow L = \sqrt{a^2+y^2} \approx a$

$\therefore m\ddot{y} = -2k(a-a_0)\frac{y}{a}$

$$\Rightarrow \boxed{m\ddot{y} = -2k\left(1-\frac{a_0}{a}\right)y}$$

In this case, ω_0 is,

$$\boxed{\omega_0 = \sqrt{\frac{2k}{m}\left(1-\frac{a_0}{a}\right)}}$$

What if a_0 is also very small? i.e, non-stretched spring length is very small.

Set $a_0 = 0 \Rightarrow$ ω_0 is same for longitudinal and transverse motion. (What does this mean physically? ☐)

Going with this, we see that x and y terms are not present in either ω_0 expression.

So, general motion,

$$\vec{r} = \left(c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right)\hat{i} + \left(c_1' e^{i\omega_1 t} + c_2' e^{-i\omega_1 t}\right)\hat{j}$$

→ Two modes of motion are completely independent (*)(*)

(Simply adding them gives general eqn of motion)

(They are not coupled)

(*) What is coupled? If in ODE of x we have y term, or vice versa

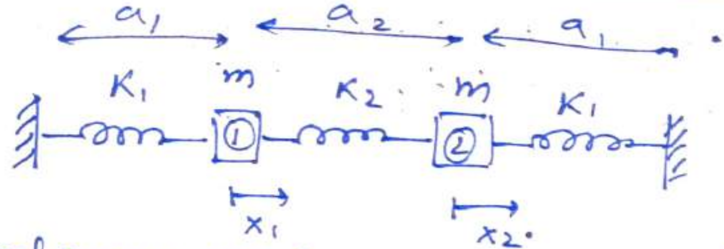
(*) These are normal modes

→ They are fixed, frozen, they do not jump from one to another.

$a\ddot{x} = by + cx$
$a'\ddot{y} = b'x + c'y$
Coupled ODEs

This will be defined better after next example.

Now,



Identical masses, massless springs.

Longitudinal motion →

Eqn of motion →

Excess force (simpler) as this is equilibrium state

If $x_1 > x_2 \Rightarrow$ Push to left.

$$m\ddot{x}_1 = -K_1x_1 - K_2(x_1 - x_2)$$

$$\cancel{m\ddot{x}_1 = -K_1x_1}$$

$$\Rightarrow m\ddot{x}_1 = -(K_1 + K_2)x_1 + K_2x_2 \quad \text{— (I)}$$

$$m\ddot{x}_2 = -K_1x_2 - (K_2)(x_2 - x_1) \rightarrow \text{Same as prev.}$$

$$\Rightarrow m\ddot{x}_2 = -(K_1 + K_2)x_2 + K_2x_1 \quad \text{— (II)}$$

Why no contribution of ~~x_1~~ spring 2 on (1)?

It affects x_2, does not affect (1) directly. Its contribution is measured by x_2 term in ODE in x_1.

If that was not the case, compression of $x_1 - x_2$ will always be true, as x_2 is static

But there is a correction due to x_2 being variable here.

That variation is how spring 2 affects (1).

Note that (I) and (II) are coupled ODEs.

Let us sum (I) and (II),

$$m(\ddot{x}_1 + \ddot{x}_2) = -(K_1 + K_2)(x_1 + x_2) + K_2(x_1 + x_2)$$

$$\Rightarrow \boxed{m(\ddot{x}_1 + \ddot{x}_2) = -K_1(x_1 + x_2)}$$

Consider $x_1 + x_2$ to be a quantity.

Now, $x_1 - x_2$,

$$m(\ddot{x}_1 - \ddot{x}_2) = -(K_1 + K_2)(x_1 - x_2) - K_2(x_1 - x_2)$$

$$\Rightarrow \boxed{m(\ddot{x}_1 - \ddot{x}_2) = -(K_1 + 2K_2)(x_1 - x_2)}$$

These ODEs are decoupled in (x_1+x_2) and (x_1-x_2)

Let us just call $x_1+x_2 = X$, $x_1-x_2 = Y$ (Note that we implicitly assume initial velocities to be the same)

If $x_1 = x_2$, Second eqn is trivially 0 all the time, the blocks will be at same dist at all times. i.e., $\dot{x}_1 = \dot{x}_2$

Let go by shifting both same distance.

$\therefore$ Eqn 1, $\cancel{2}m\ddot{x}_1 = -\cancel{2}K_1 x_1$ → In phase → $\Delta\phi = 0$.

$\Rightarrow \omega_0 = \sqrt{\frac{K_1}{m}}$

What if we push them closer, then let go?

$x_1 = -x_2$

$\Rightarrow$ First ODE is 0, trivially.

$\Rightarrow$ Second ODE,

$\cancel{2}m\ddot{x}_1 = -\cancel{2}(K_1+2K_2)x_1$

$\Rightarrow \omega_0 = \sqrt{\frac{K_1+2K_2}{m}}$ → Different ⊛ frequencies

Out of phase all the time → $\cancel{\phi}\ \Delta\phi = 180°$ all the time.

From initial condition.

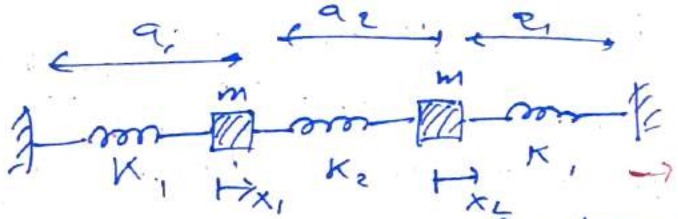
When you have multiple coupled oscillations, we have to take equations and set situations where one of the uncoupled eqns doesnot play a role. → The frequency that we find is a ~~mode~~ normal mode here. → So, if I understand, if you have coupled ODEs, you decouple them and then find initial cond such that one of these decoupled ones is zero ∀t

If the system is put in this situation it stays in that situation, i.e., ~~to~~ keeps with that frequency.

⊛ There is a more general way of doing this — we find eigen value functions. (Next class).

The solution here is one of the normal modes.

8th August 2023



→ In this case, we assume $K_1 = K_2 = K$, $a_1 = a_2 = a$

We could decouple and solve ODEs for longitudinal modes.

Assume they are moving in ~~in~~ phase.

$(x_1 + x_2) \rightarrow$ ~~const~~ inphase (If it varies)

$(x_1 - x_2) \rightarrow$ out of phase (If it varies) → Equilibrium excess force

$m\ddot{x}_1 = -K_1 x_1 - K(x_1 - x_2)$ $\quad (K_1 = K_2 = K)$

$= -2Kx_1 + Kx_2$ — (I)

$m\ddot{x}_2 = -2Kx_2 + Kx_1$ → from symmetry — (II)

Assume,

$x_1 = Ae^{i\omega t}$ → we assume freq is same (It is in a normal mode) → ω for coupled quantities in normal modes is the same for each.

$x_2 = Be^{i\omega t}$

Substitute,

$\dot{x}_1 = A\omega e^{i\omega t}$, $\ddot{x}_1 = -A\omega^2 e^{i\omega t}$

$\dot{x}_2 = B\omega e^{i\omega t}$, $\ddot{x}_2 = -B\omega^2 e^{i\omega t}$.

Substituting in (I),

$\Rightarrow -m\omega^2 A e^{i\omega t} = -2KAe^{i\omega t} + KBe^{i\omega t}$

Coeff match,

$-m\omega^2 A = -2KA + KB$ → From (I)

$\Rightarrow \boxed{(m\omega^2 - 2K)A + KB = 0}$ — (III)

(From (II)) $\boxed{(m\omega^2 - 2K)B + KA = 0}$ — (IV)

Combine them into matrix, → Simple algebraic equation, no ODE

No settling of values of x s.

We are looking for generic method so that we don't have to decouple.

$$\begin{pmatrix} m\omega^2-2K & K \\ K & m\omega^2-2K \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We look for non-trivial zeroes, i.e. A or B $\neq 0$.

$\Rightarrow$ determinant of matrix must vanish (it must be singular)

$AB = 0$

$\Rightarrow |A||B| = 0$

must be zero. is not defined, as it is 2x1 matrix

$$\Rightarrow \begin{vmatrix} m\omega^2-2K & K \\ K & m\omega^2-2K \end{vmatrix} = 0$$

$\Rightarrow (m\omega^2 - 2K)^2 - K^2 = 0$

$\cancel{\Rightarrow m\omega^2 - 2K = \sqrt{K^2}}$

$\Rightarrow m\omega^2 - 2K = \pm K \Rightarrow \omega^2 = \frac{3K}{m}, \frac{K}{m}$

$\Rightarrow \omega = \pm\sqrt{\frac{3K}{m}}, \pm\sqrt{\frac{K}{m}}$

what is the meaning of -ve ω? Clearer in travelling waves. $\rightarrow$ wave travelling to other side.

The general solution will be 4 exponentials..

$A e^{i\omega t} + B e^{-i\omega t}$

They might be different, as coeff are diff.

So, what about values of A and B?

We select a value of ω. $\rightarrow$ Say $\omega = \pm\sqrt{\frac{3K}{m}}$

$$\therefore \begin{pmatrix} m\omega^2-2K & K \\ K & m\omega^2-2K \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} K & K \\ K & K \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (A+B)\underbrace{K}_{\text{Non-zero}} = 0$$

$\Rightarrow A = -B$ (out of phase motion)

~~for $\omega = \frac{K}{}$~~ $\Rightarrow$ for $\omega = \pm\sqrt{\frac{3K}{m}}$, motion is 180° out of phase. Higher energy. ↳ Freq higher as middle spring plays a role.

for $\omega = \pm\sqrt{\frac{K}{m}} \Rightarrow (-A+B)K = 0$

$\Rightarrow A = B$ (in phase) → In this it remains at same length.

So now, how do we solve coupled oscillators? (general procedure)

1. Write down eqns of motion
2. Assume solutions with same ω (as you assume normal mode)
3. Substitute, find algebraic eqns.
4. Formulate matrix (Separating out amplitude coeffs)
5. Equate det of matrix to zero, find roots to get values of ω.
6. Select value of ω and find relation b/w amplitudes by substituting in matrix.

In bodies with 3 oscillators, we will get 3×3 matrix → 3 diff eqns (algebraic)

One of them will not be like other two

Something to do with conjugate roots?

We can solve to find,

$A = \alpha C$, $B = \beta C$ → from selection of values of ω

$(A, B, C) = (\alpha, \beta, 1)$

Don't get too rigorous here.

↳ We may have to normalize it later (unique vector). Do not worry about this now.

(*) 3 oscillator problem in tutorial this week.

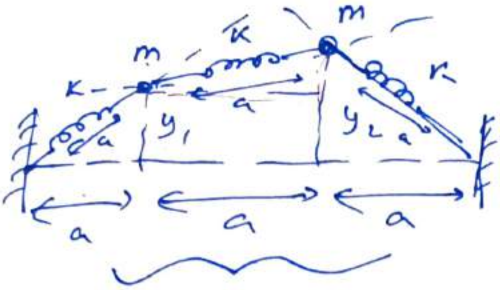
☐ Try for any longitudinal n-oscillator

☐ Do 3-oscillator.

(*) $K_1 = K_2 = K$ was taken instead of K as then the rest/eq lengths are not similar (same → otherwise eqn will just be more complicated, nothing else.

→ Tutorial will cover $K_1 \neq K_2$ (None new physics, just more computation).

II Transverse mode →



4y

Ugly.

Thanks to extreme small angles existing.

why? $y_2 > y_1$ → vertical component to that is upward

$$m\ddot{y}_1 = -K(a-a_0)\cdot\frac{y_1}{a} + T\frac{(y_2-y_1)}{a}$$

Simplicity, $K(a-a_0) = T$ (notation) → Rest length.

$$\Rightarrow \boxed{m\ddot{y}_1 = -\frac{T}{a}(2y_1 - y_2)} \quad \text{(I)}$$

Similarly,

Note: Symmetric in y_1 and y_2.

$$m\ddot{y}_2 = -\frac{Ty_2}{a} - T\frac{(y_2-y_1)}{a}$$

$$\Rightarrow \boxed{m\ddot{y}_2 = -\frac{T}{a}(2y_2 - y_1)} \quad \text{(II)}$$

Again we assume normal mode ω.

$$y_1 = Ae^{i\omega t},\quad y_2 = Be^{i\omega t}$$

$$\Rightarrow \ddot{y}_1 = -A\omega^2 e^{i\omega t},\quad \ddot{y}_2 = -B\omega^2 e^{i\omega t}$$

$$\therefore -mA\omega^2 e^{i\omega t} = -\frac{T}{a}\left(2Ae^{i\omega t} - Be^{i\omega t}\right)$$

$$\Rightarrow -mA\omega^2 = -\frac{T}{a}(2A - B)$$

$$\Rightarrow \boxed{mA\omega^2 = \frac{T}{a}(2A - B)}$$

$\Rightarrow \frac{2T}{a}A - m\omega^2 A - \frac{T}{a}B = 0$

$$\Rightarrow \boxed{\left(-\frac{2T}{a} + m\omega^2\right)A + \left(\frac{T}{a}\right)B = 0} \quad \text{— (III)}$$

~~Kac~~ $\frac{T}{a}$ acts ~~K~~ like K

Similarly,

$$\boxed{\left(m\omega^2 - \frac{2T}{a}\right)B + \left(\frac{T}{a}\right)A = 0} \quad \text{— (IV)}$$

$$\begin{bmatrix} m\omega^2 - \frac{2T}{a} & \frac{T}{a} \\ \frac{T}{a} & m\omega^2 - \frac{2T}{a} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solving $\det(\cancel{A}) = 0$

$$\left(m\omega^2 - \frac{2T}{a}\right)^2 = \left(\frac{T}{a}\right)^2$$

$$\Rightarrow m\omega^2 - \frac{2T}{a} = \pm\frac{T}{a}$$

$$\Rightarrow m\omega^2 = \frac{3T}{a}, \frac{T}{a}$$

$$\Rightarrow \omega = \sqrt{\frac{3T}{ma}}, \sqrt{\frac{T}{am}} \rightarrow \text{Solution}$$

We can use this to treat strings,
We have string of mass M, we may consider N parts.

$M = Nm \rightarrow$ mass of each part

* Read from Crawford

There are $N+1$ 'springs' b/w them.

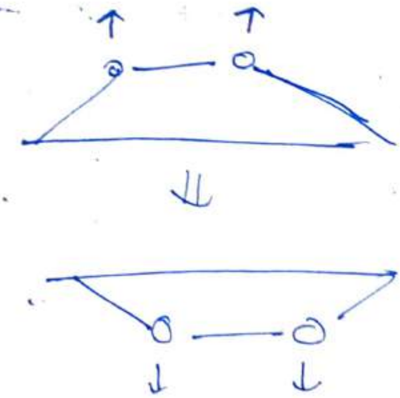
$m \propto \rho \qquad \rho = \frac{M}{(N+1)a} = \frac{Nm}{(N+1)a} \approx \frac{m}{a}$.

So, in these eqns we can see the term comes as

$\sqrt{\frac{T}{m}} \propto \sqrt{\frac{T}{m}} \rightarrow$ will be done later

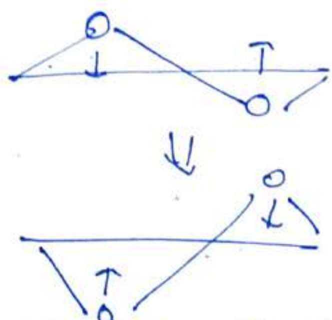
Normal modes here: →

① Inphase →



Satisfying

② Out of phase →



We are solving eigenvalue problem ⊛ ☐ Runge Ref

☐ General Solution of SHM → 11th August 2023

⊛ Systematic development of topic.

Mostly, the equation looks like,

$$m\ddot{x} = -\omega^2 x$$

↳ Dynamical variable here (Changes)

Could be θ, etc. ↳ In case of, say, pendulum.

We use a trial solution (assume),

$$x = Ae^{i\omega t} \rightarrow \text{(Rotating)}$$

Using it, ω values will be,

$$\omega = \pm \omega_0$$

Which we use to find general solution,

$$x(t) = \underbrace{Ae^{i\omega_0 t}}_{\text{+ve freq (anticlockwise)}} + \underbrace{Be^{-i\omega_0 t}}_{\text{-ve freq (clockwise)}}$$

→ Why do we need both?

for RHS to be real, A and B must be complex.

$\Rightarrow$ Expanding, $x(t) = A\cos\omega_0 t + iA\sin\omega_0 t$
$+ B\cos\omega_0 t - iB\sin\omega_0 t$

$$\Rightarrow x(t) = \underbrace{(A+B)}_{\text{Should be real}}\cos\omega_0 t + \underbrace{i(A-B)}_{\text{Should also be real}}\sin\omega_0 t$$

$$\left.\begin{aligned} A+B &= c_1 \\ i(A-B) &= c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{2}(c_1 - ic_2) \\ B &= \tfrac{1}{2}(c_1 + ic_2) \end{aligned}$$

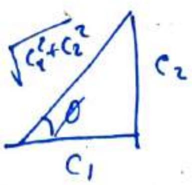
→ Reals

Solving

$$\Rightarrow x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

$$\Rightarrow x(t) = \sqrt{c_1^2+c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega_0 t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega_0 t\right]$$

We can define, for some ϕ - (*) Guess this works for combining any $c_1\cos\theta + c_2\sin\theta$ form into one $\cos\theta$ term.



$$\tan\phi = \frac{c_2}{c_1}$$

$$\Rightarrow \sin\phi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}, \quad \cos\phi = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$$

$$\Rightarrow x(t) = \sqrt{c_1^2+c_2^2}\left[\cos\phi\cos\omega_0 t + \sin\phi\,\cancel{\cos}\sin\omega_0 t\right]$$

$$\Rightarrow \boxed{x(t) = \sqrt{c_1^2+c_2^2}\cos(\omega_0 t - \phi)}$$

we don't care about c_1 and c_2, just this entire value in all. ↳ Sinusoidal

(*) We keep -ve freq because $Be^{-i\omega_0 t}$ can cancel imaginary terms of $Ae^{i\omega_0 t}$

Assuming $A\cos(\omega t + \phi)$ works well too, as this works with fitting initial values out as well.

→ In assuming solution, $x(t) = A\cos(\omega t + \phi)$ also works.

↗ hyperbolic trig □ Read up

Why? $\cos(\omega_0 t + \phi) = \dfrac{e^{i(\omega_0 t + \phi)} + e^{-i(\omega_0 t + \phi)}}{2}$

So it packages two exponentials.

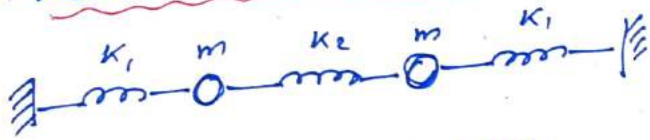
(Remember, we need 2 exp for complete ~~sol~~ general solution of one oscillator)

(*) T = Time period = $\frac{2\pi}{|\omega|}$ — -ve time period does not make physical sense.

Mod is important → T cannot be negative.

☐ Superposition of normal modes →

There can be more than one normal mode in coupled oscillators.



$$\omega^2 = \frac{K_1}{m} \text{ or } \frac{K_2 + 2K_2}{m}$$

But we don't always have only in-phase and out of phase motion. ↳ Coupled pendulum, we move one to a disp and keep the other in starting position.

As on 4th Aug, we saw the decoupled ODE as, (one of them)

$$m(\ddot{x}_1 + \ddot{x}_2) = -K_1(x_1 + x_2)$$

Looks similar to $m\ddot{x} = -\omega^2 x$. ← Using $m(\ddot{x}_1+\ddot{x}_2)$ equation.

→ Solution is, $x_1 + x_2 = c_1 \cos(\omega_1 t + \phi_1)$

$x_1 - x_2 = c_2 \cos(\omega_2 t + \phi_2)$ ↳ using the $m(\ddot{x}_1 - \ddot{x}_2)$ eqn.

$\boxed{\text{As found before}}$

We know, $\omega_1 = \sqrt{\frac{K_1}{m}}$,

$$\omega_2 = \sqrt{\frac{K_1 + 2K_2}{m}}$$

Using this, we write,

Note: These have all 4 exponential functions, 2 each $\cos(x)$.

$$x_1 = \frac{1}{2} c_1 \cos(\omega_1 t + \phi_1) + \frac{1}{2} c_2 \cos(\omega_2 t + \phi_2)$$

and, $x_2 = \frac{1}{2} c_1 (\cos \omega_1 t + \phi_1) - \frac{1}{2} c_2 \cos(\omega_2 t + \phi_2)$

⇒ These are linear superposition of motion with 2 diff freq. Two different waves superimpose.

Assignment prob →

Stationary at $t=0$

↳ Displaced by a at $t = 0$.

$\because x_1(0) = 0$

$\Rightarrow \frac{1}{2} c_1 \cos\phi_1 + \frac{1}{2} c_2 \cos\phi_2$

$\because \dot{x}(0) = 0$, (Stationary)

$\Rightarrow \frac{1}{2} c_1 \omega_1 \sin\phi_1 - \frac{1}{2} c_2 \omega_2 \sin\phi_2 = 0$

Similarly, with x_2

$x_2(0) = 0 \Rightarrow \frac{1}{2} c_1 \cos(\phi_1) - \frac{1}{2} c_2 \cos\phi_2$

$\Rightarrow \dot{x}_2(0) = 0 \Rightarrow -\frac{1}{2} c_1 \omega_1 \sin\phi_1 + \frac{1}{2} c_2 \omega_2 \sin\phi_2$

We now use these 4 eqn to find c_1, c_2, ϕ_1, ϕ_2

In ~~d~~ general,

$c_1 \sin\phi_1 = 0$ (Adding $\dot{x}_1(0) = 0$ and $\dot{x}_2(0)$)

$\hookrightarrow$ Cannot be zero $\Rightarrow$ Otherwise it would be out of phase motion at $t=0$

$\Rightarrow$ Non-physical

$\Rightarrow c \neq 0$.

$\Rightarrow \sin\phi = 0 \Rightarrow \boxed{\phi = n\pi,\ n \in \mathbb{Z}^+}$

Subtracting rel eqn,

$c_2 \sin\phi_2 = 0$

By same argument,

$\boxed{\phi_2 = n\pi,\ n \in \mathbb{Z}^+}$

$\circledast\ c_1 + c_2 = 0$

$\circledast\ \frac{1}{2}(c_1 - c_2) = 2a$ $\Big\} \rightarrow$ Also from $x(t)$ 1.

~~Sotati~~ Solving,

$c_1 = a,\ c_2 = -a$.

So, our final solution is,

$$x_1 = \frac{1}{2} a \left(\cos(\omega_1 t) - \cos(\omega_2 t)\right)$$

$$\Rightarrow \boxed{x_1 = \cancel{\tfrac{1}{2}} a \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_2 - \omega_1}{2} t\right)}$$

$$x_2 = \frac{1}{2} a \left(\cos\omega_1 t + \cos\omega_2 t\right)$$

$$\Rightarrow \boxed{x_2 = \cancel{\tfrac{1}{2}} a \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_2 - \omega_1}{2} t\right)}$$

Note that the envelop terms (trig func with lower freq $\omega_2 - \omega_1$) are 180° out of phase.

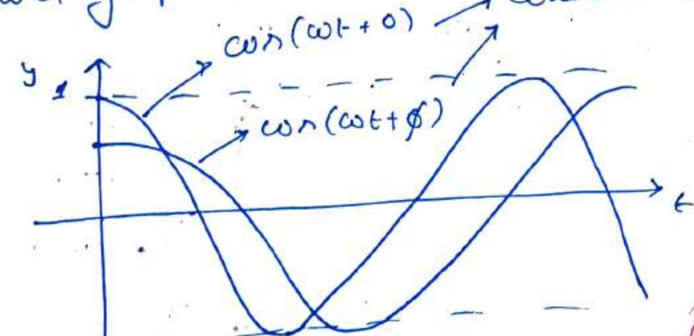
Like double pendulum. Same eqn were derived there.

(*) Coherent sources have fixed phase differences. (Def)

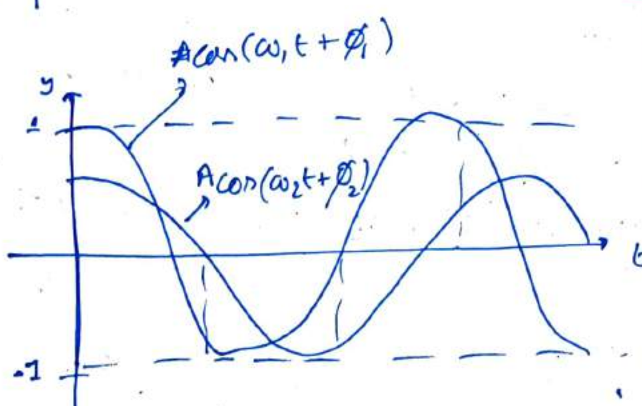
SHM gives solutions that are coherent, as $\phi_1(t)$ and $\phi_2(t)$ does not change with time.

Now, we try to plot,

Coherent (shifted by ϕ)



Now,

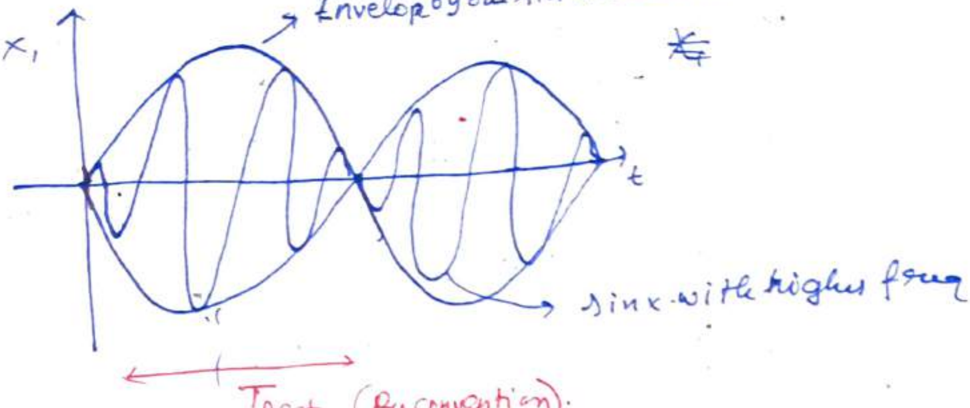


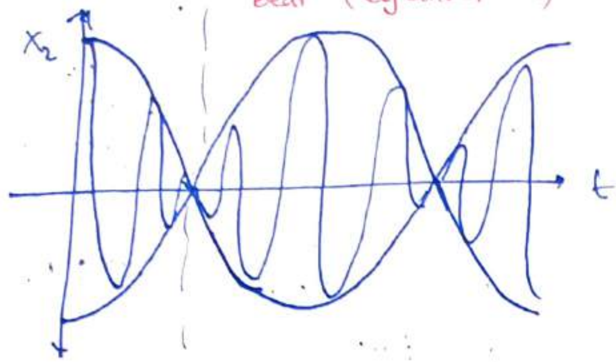
(*) Constant term, i.e. term not compounded with time must remain constant for the waves to be coherent

$$\Delta\phi : \text{Phase differ} = \underbrace{(\omega_1 - \omega_2)t}_{\text{We don't care about this}} + \underbrace{(\phi_1 - \phi_2)}_{\text{We care about the non temporal part}}$$

(*) If this is const, the waves are coherent

Now we try to plot the solution to x_1 and x_2,





While x_1 is oscillating at full amp,
x_2 is stable (Double pendulum)

~~Beat~~ Beat freq $\longrightarrow \frac{1}{2} \dfrac{2\pi}{\left(\frac{\omega_2-\omega_1}{2}\right)}$ → Half of the time freq by longer time period (envelope)

$= \dfrac{2\pi}{\omega_2-\omega_1}$

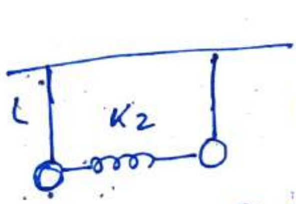
(*) Discussion of the non-linear behaviour of the double pendulum

⇒ He mentioned that the 'spring' is non-linear,
and $F = cd + c'd^3 + \dots$ (As we discussed with Bessel)

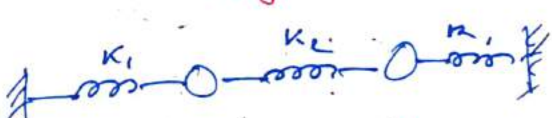
Next, we will be studying strings and beaded strings.

Find out why. ☐

Now,



and

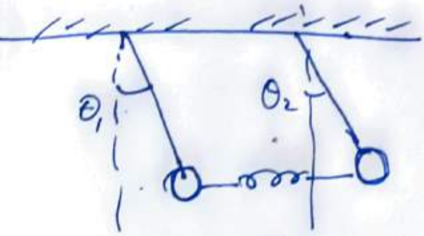


are similar,

$\omega = \sqrt{\frac{g}{l}}$ ⟵ are similar ⟶ $\omega = \sqrt{\frac{K_1}{m}}$

☐ Solve and see for this case

How?



$$ml^2\ddot{\theta}_1 = -mgl\sin\theta_1 - k_c l(\sin\theta_1 - \sin\theta_2)\, l\cos\theta_1$$

$\downarrow$

Same as inphase

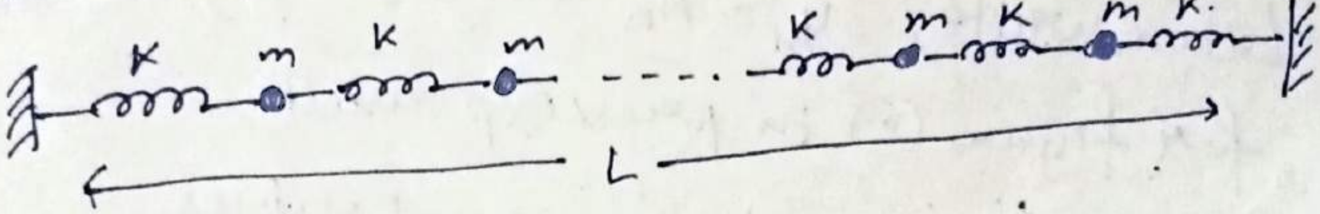
On small angle approximations,
we can solve this to get similar solutions.

II Point.

15th August 2023

Normal modes of a beaded string →

We model it as,

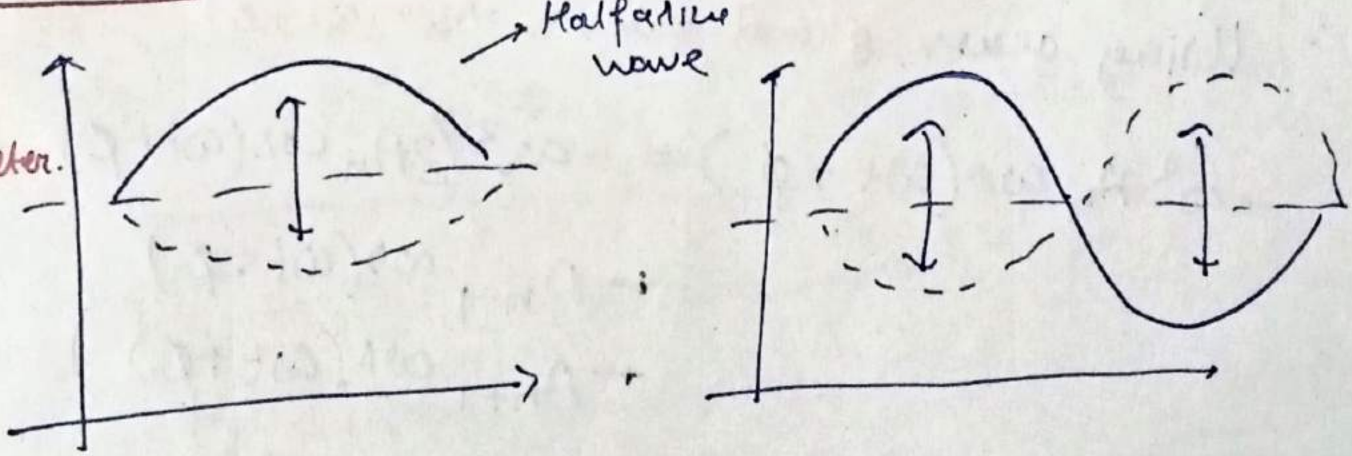


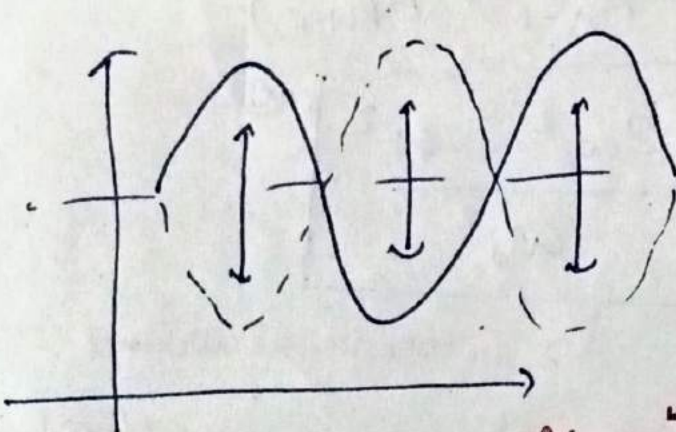
We cannot do this the usual way.

A beaded string is a 'lumped' version of a continuous string — an approximation.

Experiments were done, and the modes found were like →

Typical sonometer.



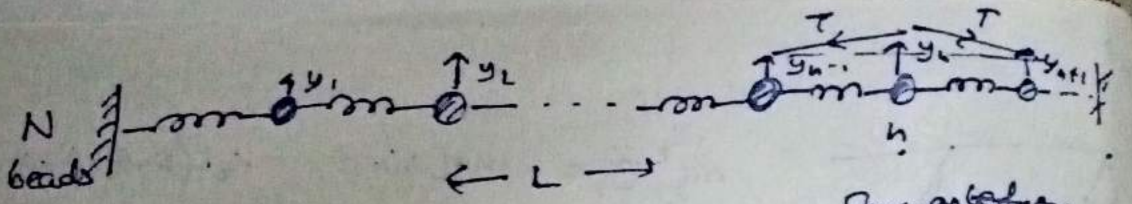


We now formulate math.

⊛ each bead has fixed temporal phase relation ~~[illegible]~~ to any other!

— But amplitudes are diff.

At some t, they have fixed position w.r.t. other.



Same as before

Under small angle approximation,

$$m\ddot{y}_n = -T\left(\frac{y_n - y_{n-1}}{a}\right) - T\left(\frac{y_n - y_{n+1}}{a}\right)$$

Same notation as longitudinal case before.

$$\Rightarrow \ddot{y}_n = -\frac{T}{ma}\left(2y_n - y_{n-1} - y_{n+1}\right)$$

We define, $\boxed{\omega_0^2 = \frac{T}{ma}}$ → We just take whatever is out there as a const factor and call it ω_0^2

$$\Rightarrow \boxed{\ddot{y}_n = -\omega_0^2\left(2y_n - y_{n-1} - y_{n+1}\right)}$$

Let us write, $y_n = A_n \cos(\omega t + \phi)$ → position func of n^{th} bead.

In prev exp results,
for figure (2) in normal values,
upto $\frac{N}{2}$ we can have normal values,
but after that we have to have a -ve sign
to denote that they are always on opp
ends.

[?] Why do we take exp guess sometimes and we don't use exp guess for cos sometimes?

↗ simplification i.e.

$\therefore$ Using guess, usual guess, $y_n = A_n\cos(\omega t + \phi)$

$$-\omega^2 A_n \cos(\omega t + \phi) = -\omega_0^2\big(2A_n\cos(\omega t + \phi) - A_{n-1}\cos(\omega t+\phi) - A_{n+1}\cos(\omega t + \phi)\big)$$

$$\Rightarrow \omega^2 A_n = \omega_0^2\left(2A_n - A_{n-1} - A_{n+1}\right)$$

(*)

$$\Rightarrow \boxed{\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}}$$

↳ Interesting recurrence.

LHS is a function of n.
How do we show that it holds? → It is meant that how do we show that ω is not $f(n)$ as we presumed for a mode?

We start with trial solution → guessing A_n.

Since the expt data looks like sine wave,

$\boxed{A_n = C\sin(n\theta)}$ Assumed function

→ I checked it numerically, and the only ~~...~~ explicit assumption required is that $\frac{2\omega_0^2-\omega^2}{\omega_0^2}$ is not a function of n. The recurrence naturally gives a sinusoid when computed.

∴ LHS of prev relation,

Why not cos? (Think of boundary cond of strings)

$$\frac{C\sin(n-1)\theta + C\sin(n+1)\theta}{C\sin n\theta}$$

$$= \frac{2\cancel{\sin n\theta}\cos\theta}{\cancel{\sin n\theta}} = 2\cos\theta$$

So, for $A_n = c\sin(n\theta)$,

$$2\cos\theta = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$$

→ Indep of n (As found)

Are there other solutions? — we are not formally doing it atm, but ~~...~~ there are in fact none other..

→ ☐ Check how we can formally verify that there are no other solutions that fit A_n.

$$\omega^2 = 2\omega_0^2(1-\cos\theta)$$

$$\Rightarrow \omega^2 = 2\omega_0^2\cdot\left(2\sin^2\frac{\theta}{2}\right)$$

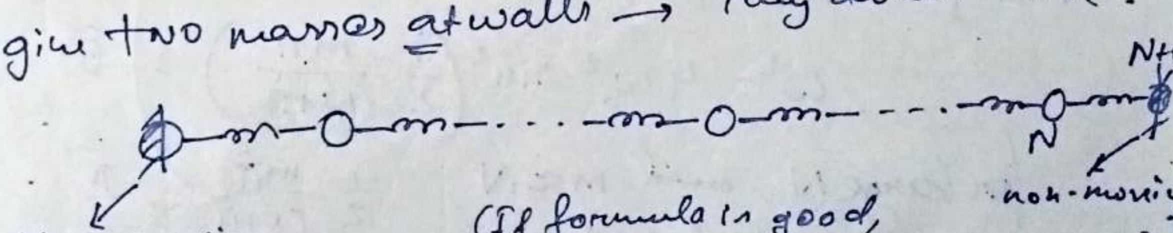
$$\Rightarrow \boxed{\omega^2 = 4\omega_0^2\sin^2\frac{\theta}{2}}$$

→ Upper limit of ~~θ~~ ω

→ This sets an upper lim to ω as sin is bounded

What is θ? Is it just a parameter?

Imagine two masses at walls → They do not move.



(If formula is good, it must hold at end points)

Now, $A_{N+1} = 0 = C\sin(N+1)\theta$

$\Rightarrow (N+1)\theta = m\pi, \quad m\in\mathbb{Z}$

$\Rightarrow \boxed{\theta = \frac{m}{(N+1)}\pi}$ → θ 'quantises' or encodes mode information.

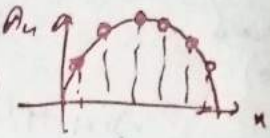
∴ $\boxed{A_n = C \sin \frac{n m \pi}{(N+1)}}$ ⊛ IMP

Say we set ~~m =~~ m = 1

Since $n \neq (N+1)$ ever, we never have sin term go to zero. → All beads are on same side at some t.

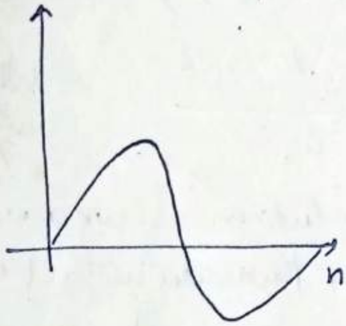
$0 < \frac{n}{N+1} < 1$ ⇒ We get a unique sine wave like

fig ① expt solution. →



What if m = 2

$\sin \frac{2n\pi}{N+1}$ goes somewhat like →



Like fig ②

Note, higher the value of m, more the nodes

Kind of predicts normal modes → m is normal mode index ⊛

~~Wh~~ What is max value of m? → We did bound ~~An~~ A_n, but what about m?

We can get from

$$\omega^2 = 4\omega_0^2 \sin^2 \frac{\theta}{2}$$

Putting in value of θ,

$$\omega^2 = 4\omega_0^2 \sin^2\left(\frac{1}{2}\frac{m\pi}{(N+1)}\right) \quad \text{— ①}$$

for large N, and $m \approx N$, $\frac{1}{2}\frac{m\pi}{(N+1)} \approx \frac{\pi}{2}$

↳ many beads, nearly continuous string

→ giving in max $\sin^2\left(\frac{\pi}{2}\right) = 1$ → Verify I ⊛

⇒ $\omega^2 \approx 4\omega_0^2$

I mean, $\lim_{N\to\infty}\left(\frac{m}{N+1}\right)$, $m \approx N$. ⇒ $\lim_{N\to\infty}\frac{N}{N+1} \overset{?}{=} 0$, but we know limits are stupid

$\omega_0 \rightarrow$ Characteristic freq of system (Characterises system)

'Eigen freq' $\Leftrightarrow$ Normal mode freq.

$\rightarrow$ Again with the 'eigen'

So, the expression right now,

$$\omega^2 = 4\omega_0^2 \sin^2\left(\frac{1}{2}\frac{m\pi}{(N+1)}\right)$$

$$A = C\sin\frac{nm\pi}{(N+1)}$$

(Multiplying num and denom with a)

$$\Rightarrow A = C\sin\left\{\frac{m\pi\,(na)}{(N+1)a}\right\}$$

$na \rightarrow$ Horizontal position of bead. $\Rightarrow x_n$

$(N+1)a \rightarrow L =$ Length of string.

$$\Rightarrow A = C\sin\left(\left(\frac{m\pi}{L}\right)x_n\right)$$

$k = \frac{m\pi}{L}$ = Angular Wave number (No. of rad per unit dist)

$$\Rightarrow A = C\sin(k x_n)$$

$\hookrightarrow$ Why? Normally, $k = \frac{2\pi}{\lambda} = \frac{m\pi}{L}$

$\Rightarrow \boxed{\lambda = \frac{2L}{m}}$ (*)

In physics, you often have conjugate variables $\rightarrow$ products give physically meaningful quantities.

We define, $k = \frac{2\pi}{\lambda}$

$2\pi \rightarrow$ Dimension of radian (To make calc simpler)

$\lambda \rightarrow$ wavelength of sinusoid

We could just choose $k = \frac{1}{\lambda}$, does not matter.

But when $x_n = \lambda$, $kx_n = 1$

But $\sin 1$ isn't what we need

$\hookrightarrow$ sin function is what necessitates that 2π factor

(*)

$$\Rightarrow \boxed{A = C\sin(kx_n)}$$

$$\therefore k = \frac{m\pi}{L} = \frac{2\pi}{\lambda} \Rightarrow \lambda = (2\pi)\left(\frac{L}{m\pi}\right)$$

$$\Rightarrow \boxed{\lambda = \frac{2L}{m}}$$

(*) IMP

(*) $\rightarrow$ Set $m=1$, $\lambda = 2L$ (True)

Set $m = 2$, $\lambda = L$ (True)

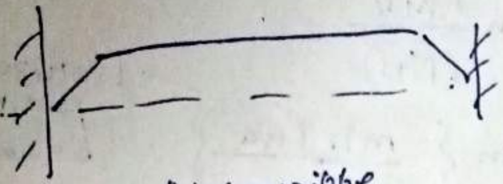
etc...

Does λ have a limit (lower?)?

→ makes no sense for $\lambda \leq a$. (Less than dist b/w two beads)

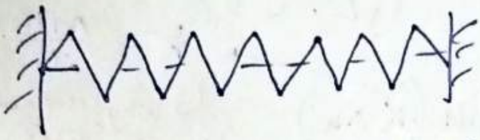
If $\lambda = a \Rightarrow$ All beads move uniformly. → All beads have the exact same phase.

(Non-physical)



Not possible

So, $\lambda = 2a$ is allowed,



every alternate bead is out of phase. → Physically intuitive

Alt approach → Plot, $\omega^2 = 4\omega_0^2 \sin^2\left(\frac{1}{2}\frac{m\pi}{(N+1)}\right)$ as f(m)

→ has a max value, $\omega^2 = 4\omega_0^2$

as $\frac{1}{2}\frac{m\pi}{(N+1)} \approx \frac{\pi}{2}$

or $m = (N+1)$

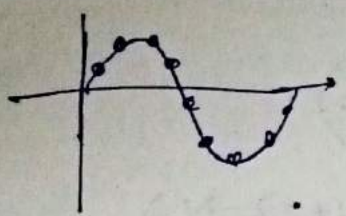
Substitute for λ expression, $a(N+1)$

$$\lambda = \frac{2L}{m} \Rightarrow \lambda = \frac{2L}{(N+1)}$$

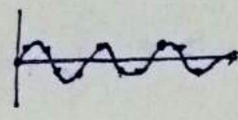
$$\Rightarrow \lambda = \frac{2a\cancel{(N+1)}}{\cancel{(N+1)}} \Rightarrow \boxed{\frac{\lambda}{2} = a}$$

(Same result)

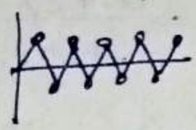
⊛ Brief remark on wave nature of discrete particles →



for lower λ

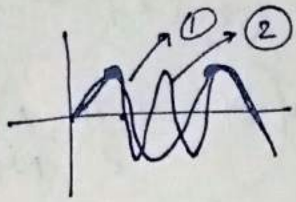


Even lower λ



→ Anything smaller doesnot mean anything, → physical, that is.

as for



→ Infinite waves explain it

⟹ We conventionally choose the largest λ here. ⊛

22nd August 2023

We recount the formulae →

$$\omega^2 = 4\omega_0^2 \sin^2 \frac{1}{2} ka$$

$$= 4\omega_0^2 \sin^2 \frac{1}{2} \frac{m\pi}{(N+1)}$$

$$k = \frac{m\pi}{L} = \frac{m\pi}{(N+1)a}$$

⊛ V.Imp

$$\boxed{\omega = 2\omega_0 \sin\left[\frac{\pi}{2}\left(\frac{m}{N+1}\right)\right]}$$

$$\boxed{\omega_0 = \sqrt{\frac{T}{ma}}}$$

→ m can go to infinity, no real bounds there, but there should be only N normal modes

↓

We see that any value of m over N gives meaningful information (doesnot)

So, $\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{2}\frac{1}{N+1}\right)$

$\omega_2 = 2\omega_0 \sin\left(\frac{\pi}{2}\frac{2}{N+1}\right)$

$\vdots$

$\omega_n = 2\omega_0 \sin\left(\frac{\pi}{2} \cdot \frac{N}{N+1}\right)$

$\omega_{N+1} = 2\omega_0 \sin\left(\frac{\pi}{2} \cdot \frac{N+1}{N+1}\right) = 2\omega_0$ → Highest possible freq.

So, what happens with ω_{N+2} ?

$$\omega_{N+2} = 2\omega_0 \sin\left(\frac{\pi}{2} \cdot \frac{N+2}{N+1}\right)$$

$$\Rightarrow \omega_{N+2} = 2\omega_0 \sin\left(\frac{\pi}{2} \frac{(2N+2)-N}{N+1}\right)$$

$$\Rightarrow \omega_{N+2} = 2\omega_0 \sin\left(\frac{\pi}{2}(2) - \frac{\pi}{2}\frac{N}{N+1}\right)$$

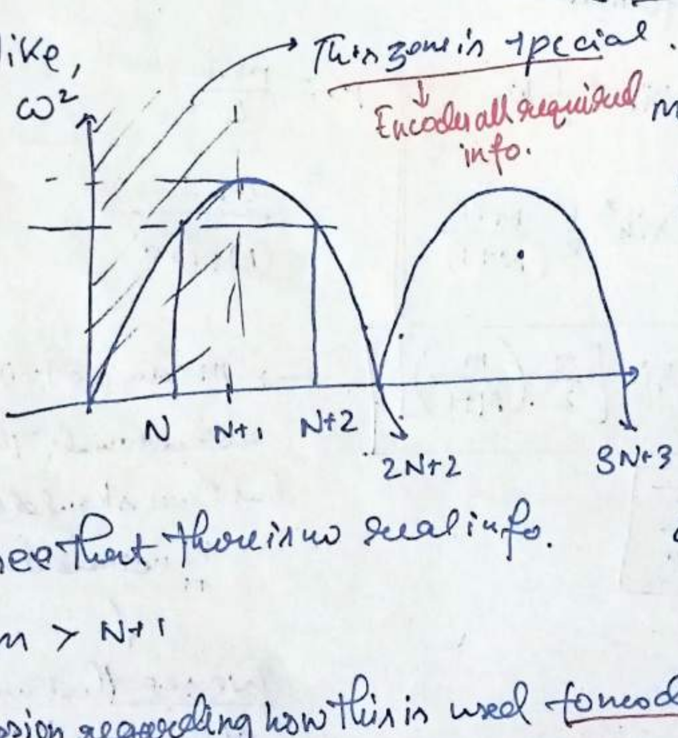
$$\Rightarrow \omega_{N+2} = 2\omega_0 \sin\left(\pi - \frac{\pi}{2}\frac{N}{N+1}\right)$$

$$\Rightarrow \omega_{N+2} = 2\omega_0 \sin\left(\frac{\pi}{2}\frac{N}{N+1}\right)$$

$$\Rightarrow \boxed{\omega_{N+2} = \omega_N}$$ → Obvious — ω_N is a periodic function of N.

You can prove → $\boxed{\omega_{N+3} = \omega_{N-1}}$ ☐ Prove ↳ ☐ Do

It is like,



makes sense, as $\omega^2 \propto \sin^2$ (some)

Easy to see that there is no real info. and so on.
for $m > N+1$

(*) Discussion regarding how this is used to model lattice energies using lennard-jones potentials → ☐ Read (*)

(*) Relations b/w freq (ω) and wavenumber (k) are called dispersion equations → useful in physics.

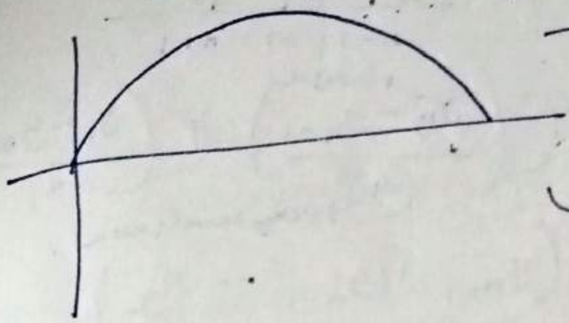
↓

V. Imp (*)

We derived expression for amplitude of n^{th} bead,

$$\boxed{A_n^{(m)} = C \sin\left(\frac{nm\pi}{N+1}\right)} \circledast$$

for, $A_n^{(1)} = C \cdot \sin\left(\frac{n\pi}{(N+1)}\right)$



→ Always +ve for $1 \leq n \leq N+1$

↳ first normal mode, agrees with expt.

What for,

$$A_n^{(2)} = C \sin\left(\frac{2n\pi}{N+1}\right)$$

$\vdots$

$$A_n^{(N)} = C \sin\left(\frac{Nn\pi}{(N+1)}\right) \quad !$$

But note,

$$\boxed{A_n^{(N+1)} = 0 \quad \forall n}$$

↳ All beads become nodes themselves.

→ So this $\omega = 2\omega_0$ thing is never really seen.

We can also see that,

$$\boxed{A_n^{(N+2)} = A_n^{(N)}}$$ ← ☐ Verify.

→ Again, a periodic function of N. (m)

$$A_n^{(N)} = C \sin\left[\frac{(N+1)-1}{(N+1)} n\pi\right]$$

$$A_n^{(N)} = C \sin\left[n\pi - \frac{n\pi}{N+1}\right]$$

$$\Rightarrow A_n^{(N)} = C \ast \underbrace{((-1)^{n-1})}_{} \sin\left(\frac{n\pi}{N+1}\right)$$

zig-zag pattern predicted.

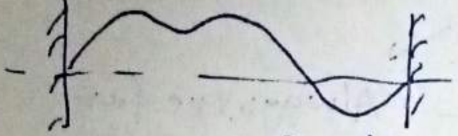
⊛ Some revision req.

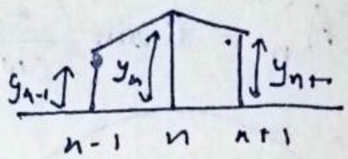
☐ Re-do trig formulae.

[] Try longitudinal mode for beaded. (Same result expected).

Assignment 2

O Continuous string → Smaller, closer beads (approx.





Small. large.

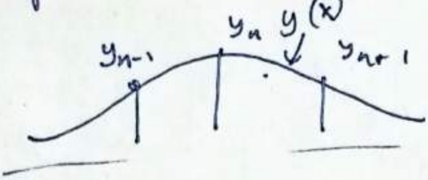
$$m\ddot{y}_n = -T\left(\frac{y_n - y_{n-1}}{a}\right) - T\left(\frac{y_n - y_{n+1}}{a}\right)$$

very small no.

Derived before $\to m\ddot{y}_n = \frac{-T}{a}\left(y_{n+1} + y_{n-1} - 2y_n\right)$

Usual

Since it is continuous, we can consider that we are choosing discrete points on a continuous curve.



We know, $\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$

We use taylor series expansion of $y(x)$ to evaluate $y(x+h)$,

[] Read more about Taylor/ Mclaurin approx.

$$\boxed{y(x+h) = y(x) + \frac{1}{1!}\frac{dy}{dx}\Big|_h h + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_h h + \dots O[h^3]}$$

Now,

$$y(x-h) = y(x) - \frac{1}{1!}\frac{dy}{dx}\Big|_{x=h} h + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x=h} h + \dots O[h^3]$$

Adding,

$$\lim_{h\to 0} \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} = \lim_{h\to 0}\left[\frac{d^2y}{dx^2} + \frac{O[h^4]}{h^2}\right]$$

goes to zero.

$$\Rightarrow \boxed{\lim_{h\to 0} \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} = \frac{d^2y}{dx^2}}$$

Nice Approx worth remembering

Expression for second derivative

We take snapshot of string at some time t.

na → of course.

$$m\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{a}\left[y((n+1)a,t) + y((n-1)a,t) - 2y(na,t)\right]$$

In limit $a \to 0$,

$$m\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{a}\left(a^2 \cdot \frac{\partial^2 y(x,t)}{\partial x^2}\right)$$

→ Why does this work? Because for continuous strings $a \to 0$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\left(\frac{m}{a}\right)} \cdot \frac{\partial^2 y}{\partial x^2}}$$

V. V. IMP

Wave equation

$$\left[\frac{T}{(m/a)}\right] = L^2T^{-2} = \boxed{[v^2]}$$

Characteristic ~~wave~~ phase velocity → Definition?

?

Indicative of how fast a wave 'travels' (i.e, carries information)

T depends on ~~the~~ normal mode, as higher modes have more tension.

So far we have been looking at discrete wave eqn.

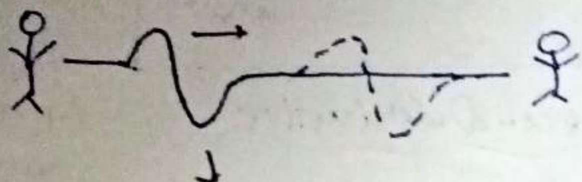
25th August 2.

Wave equation →

(*) Commit to memory completely.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

We discuss properties of this equation.



What propagates? → We refer to a pattern moving

What does that mean?

We refer to the energy (disturbance/pattern it causes) to propagate → So really, energy is what can be said to 'move'.

How do we get this information from wave eqn?

It has two eq. solutions (types) → → ☐ How do we know the nature of solutions?

① Standing wave
② Propagating end

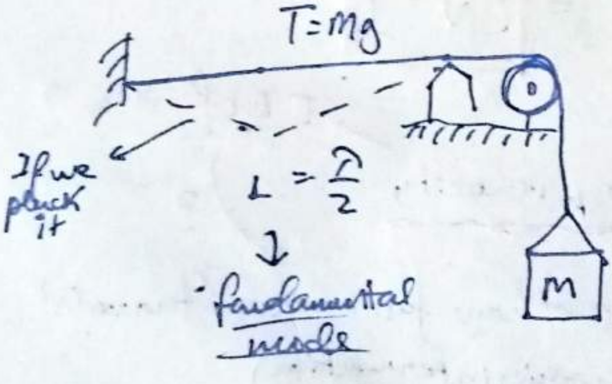
→ String fixed at end (Standing)
or
String free at one (travelling)

↓

Boundary conditions

↳ Define nature of solution.

We start with fixed string



Sonometer setup.

What is the solution that we expect here?

$y = c f(x) g(t)$, → Completely separated

at $t=0$, $y = c f(x)$ (No effect of t)

→ Triangular due to pluck.

Why are we assuming that solution is separable?

⇒ Guided by experiment / intuition. → That is, if it must hold for all time, this time must be same

So, proposed solution for wave eqn for bounded standing wave,

IMP → This is how you solve for wave eqn.

$$y = c f(x) g(t)$$

$$\frac{\partial^2 y}{\partial t^2} = c f(x) g''(t), \quad \frac{\partial y}{\partial x} = c f''(x) g(t)$$

$$\therefore \cancel{c} f(x) \frac{d^2 g}{dt^2} = v^2 \cancel{c} \frac{d^2 f}{dx^2} g(t).$$

$$\Rightarrow \underbrace{\frac{1}{g(t)} \frac{d^2 g}{dt^2}}_{\text{func of time}} = \underbrace{\frac{v^2}{f(x)} \frac{d^2 f}{dx^2}}_{\text{func of space}} = \text{Constant}$$

for this to be true

⇓

(If this is to hold, it needs to be a constant that this is equal to (space and time are independent variables))

$$\therefore \frac{1}{g} \frac{d^2 g}{dt^2} = -\omega^2$$ → Assume constant.

$$\Rightarrow \boxed{g(t) = A\cos(\omega t + \phi)}$$ → $\frac{1}{g} \cdot \ddot{g} = \lambda \Rightarrow \ddot{g} - \lambda g = 0$ ⇒ D.o by operator.

Also,

$$\frac{v^2}{f(x)} \frac{d^2 f(x)}{dx^2} = -\omega^2 \Rightarrow \boxed{f(x) = B\cos\left(\frac{\omega}{v} x + \phi_i\right)}$$

$f(0) = 0$, (node fixed) ~~for~~ glor (Boundary condition)

$f(0) = 0$

$\Rightarrow \cos(\phi_1) = 0 \Rightarrow \boxed{\phi_1 = 2n\pi + \frac{\pi}{2}}$

(*) Note again – this would not hold for travelling waves due to assumption → Boundary conditions are not this.

I Why must the constant be a purely negative number?
↳ otherwise of the time part will give complex solutions

$\therefore f(x) = B\cos\left(\frac{\omega}{v}x + \frac{\pi}{2}\right) = B\sin\left(\frac{\omega}{v}x\right)$

↓ It is a quality of life update.

$f(L) = 0$ (Boundary)

$\Rightarrow \sin\left(\frac{\omega}{v}L\right) = 0 \Rightarrow \frac{\omega}{v}L = m\pi$

$\Rightarrow \boxed{\omega = \left(\frac{m\pi}{L}\right) v}$

→ Can be expressed as wavenumber.

$\therefore K = \frac{2\pi}{\lambda}$ → Angular wavenumber.

(*) → IMP ← (*)

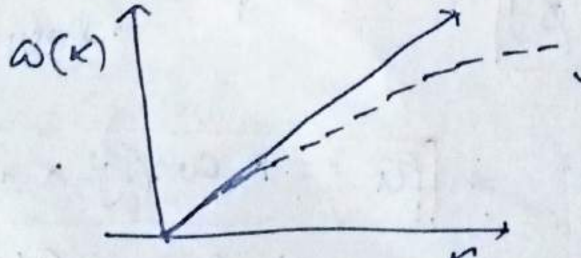
$\boxed{K = \frac{\omega}{v}} \Rightarrow \boxed{K = \frac{m\pi}{L}}$

Constant (*) for setup. (Remember, it is not due to integration of ODE)

$\Rightarrow \boxed{\omega \propto K}$ → dispersion relation

But there are cases where this is not true → Need to calculate dispersion relation in that case.

for beaded string,



→ Linear case (special)

→ Beaded string

Try to understand what this entire business of dispersion relations is

In such cases where medium is not linear → ☐ Example?

$\omega(k) = \alpha k + \beta k^2 + \dots$ → Dispersion relation ⊛

$\frac{\omega}{k} = \alpha + \beta k + \dots$

⊛ This is important in atomic physics

Since k changes uniformly, as $\boxed{\frac{2\pi}{\lambda} = \frac{n}{\text{some integers}}}$

$\boxed{\lambda = nL}$, $n \in \mathbb{Z}$ ⇒ within exp control (parameter)

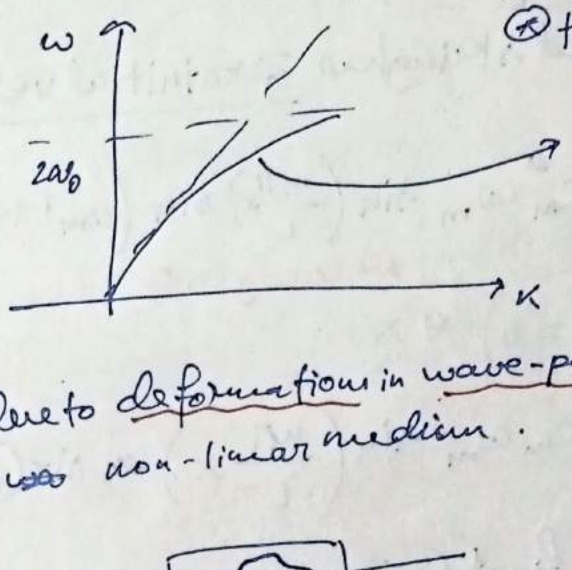
↳ L is fixed, we control n.

But ω may not change proportionally with it.

In beaded string,

$$\omega = 2\omega_0 \sin\left(\frac{1}{2}ka\right) = 2\omega_0 \sin\left(\frac{1}{2}\frac{m\pi}{L}a\right)$$

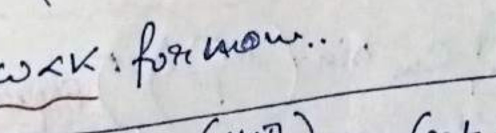
∴ We know $2\omega_0 \geq \omega$ is max possible → sinusoidal relation b/w ω and k.



⊛ Higher k can mean lower speed.

This is due to deformation in wave-packet during propagation in non-linear medium. → ??

Later



(We will revisit later in course)

Assume $\omega \propto k$ for now.. ⊛ IMP

∴ $\boxed{y = C\sin\left(\frac{m\pi}{L}x\right)\cos(\omega t + \phi)}$ m^{th} normal mode.

But what if we want to represent general motion?

→ Linear combination of all normal mode.

$$\boxed{y = \sum_m C_m \sin\left(\frac{m\pi}{L}x\right)\cos(\omega_m t + \phi_m) \qquad \omega_m = \frac{m\pi}{L}v}$$

(*)

Differs for normal mode.

↓

[] Flow? Show.

Set $t=0$ to find pluck situation.

$$y = \sum_m C_m \sin\left(\frac{m\pi}{L}x\right)\cos(\phi_m)$$

absorb.

$$\Rightarrow y = \sum_m C_m' \sin\left(\frac{m\pi}{L}x\right)$$

↓

We need to find how C_m' Looks.

[] Try to understand the flow of logic here.

(*) Note → Plucked string has zero initial velocity

$$\therefore \frac{dy}{dt} = -\sum_m C_m \omega_m \sin\left(\frac{m\pi}{L}x\right)\sin(\omega_m t + \phi_m)$$

At $t=0$, $\frac{dy}{dt} = 0$ $\forall x$ → velocity is 0

$$\Rightarrow 0 = -\sum_m C_m \omega_m \sin\left(\frac{m\pi}{L}x\right)\sin(\phi_m)$$

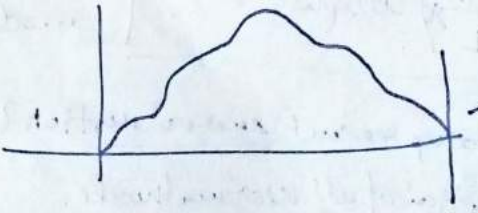
We are then to find C_m expression → If we know that we know soln for all time.

$\boxed{\sin(\phi_m) = 0}$ is possible, as $C_m = 0 = \omega_m$ is nonphysical

$$\text{So, } y(x,0) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi}{L}x\right)$$

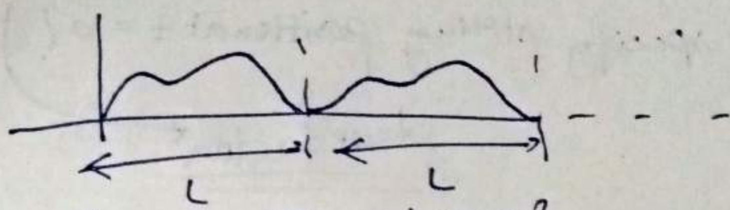
Expressible as

This is what we are

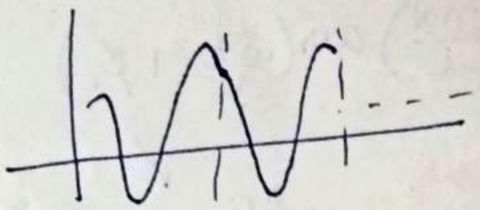


Boundary condition satisfied.

Fourier observed that such representation of ~~repea~~ any repeated patterns is possible.



What if end points are not zero?



→ We need a constant, and cosine function too

☐ Only when both ends are not zero?

(*) for any piecewise continuous repeating function,

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{L}x\right) + \sum_{m=0}^{\infty} b_m \cos\left(\frac{m\pi}{L}x\right)$$

Fourier Series

(*) If it does not repeat, it can still be represented as continuous integral (later).

☐ Look into fourier series representation of non-repeating functions.

29th Aug 2[illegible]

$$y(x,t) = c\sin\left(\frac{\pi x}{l}\right)\cos(\omega_1 t + \phi_1)$$

How do we specify string position at $t = 0$?

phase factor

Say for,

$$y(x,t) = c\sin\left(\frac{2\pi x}{l}\right)\cos(\omega_2 t + \phi_2)$$

Also encodes same information — initial po[illegible]

* it is the only term in temporal phase that survives at $t=0$

(*) All normal modes have independent phase factor.

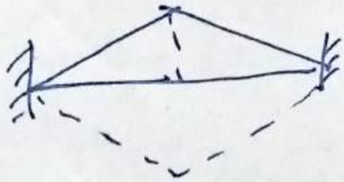
↳ What is meant?

When we have motions that are a mixture of normal modes — The phase differences linearly combine (*)

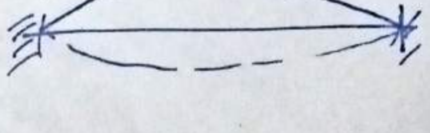
i.e, if ϕ_1 and ϕ_2 are diff in each ~~comb~~ comb but the other parameters are the same, then the combinations are same but only lag (each w.r.t each other.

Fourier Series →

Say we pluck a string in the middle,

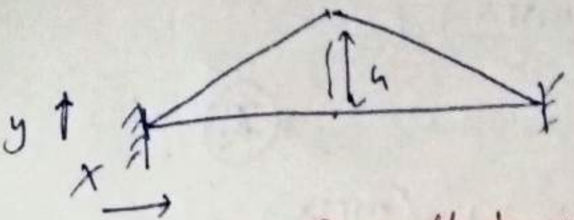


But this does not look like a normal mode



Smooth sine curve

We do have these in the pluck — we ~~f~~ find them by Fourier decomposing the pluck.



$$y(x,0) = \begin{cases} \frac{h}{(L/2)}, & x < \frac{1}{2}L \\ -\frac{h}{(L/2)} + 2h, & x > \frac{L}{2},\ x < L \end{cases}$$

Reasonable to assume that the pluck is made up of combination of linear modes.

Now,

$$y(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{\cancel{m}\, n\pi x}{L}\right) \longrightarrow \text{Normal modes}$$

Notice that all the cosines are removed — so what encodes temporal factors / phase factor?

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \phi_n)}$$

(General expression) ↳ we just care about forming x dep part — t dep part remains same.

Differentiating,

$$\dot{y}(x,t) = -\sum_{n=1}^{\infty} C_n\, \omega_n \sin\left(\frac{n\pi \cancel{x}}{L}\right)\sin(\omega_n t + \phi_n)$$

Now, $\dot{y}(x,t) = 0$ at $t = 0$.

This will only happen if $\boxed{\phi_n = 0\ \forall n.}$

We find C_n →

$$\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad \text{we compute.}$$

What are we doing? If we have a function that can be expressed as sum of orthogonal function, we multiply with function for coeff we wish to find and integrate.

$$\therefore \int_0^L y(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \underbrace{\int_0^{L/2} \frac{2h}{L} x \sin\left(\frac{m\pi x}{L}\right) dx}_{I_1}$$

$$+ \underbrace{\int_{L/2}^{L} \left(2h - \frac{2h}{L}x\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{I_2}$$

for I_1,

$$I_1 = \int_0^{L/2} \frac{2h}{L} x \sin\left(\frac{m\pi x}{L}\right) dx$$

$\boxed{\text{Usual Fourier stuff - no comments}}$

By parts,

$$I_1 = \frac{2h}{L}\left\{ x \int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right) dx - \int_0^{L/2} \left(\frac{-\cos\left(\frac{m\pi x}{L}\right)}{\frac{m\pi}{L}} \right) dx \right\}$$

$$\Rightarrow I_1 = \frac{-hL}{m\pi} \cos\left(\frac{m\pi}{2}\right) + \frac{2hL}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right)$$

and similarly with I_2, (we skip because I am lazy)

$$\therefore \int_0^L y(x) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{4hL}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right)$$

On the other hand,

$$\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\frac{c_n}{2}\underbrace{\int_0^L 2\cdot\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{I}$$

Now, if $n \neq m$, I vanishes. $\rightarrow$ orthogonality

~~In the~~ In case $m = n$, $I = L$

$$\Rightarrow \int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\frac{c_n}{2}(L)\,\delta_{n,m}$$

Kronecker delta

$$\delta_{pe} = \begin{cases} 1, & p = e \\ 0, & p \neq e \end{cases}$$

$$\Rightarrow \int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{c_m}{2}L$$

$$\therefore \boxed{c_m = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$ Imp, remember

So when you pluck, write down $y(x)$ at $t=0$, do this to find $y(x)$.

Here, we know (computed last page)

$$\Rightarrow \boxed{c_m = \frac{8h}{(m\pi)^2}\sin\left(\frac{m\pi}{2}\right)}$$

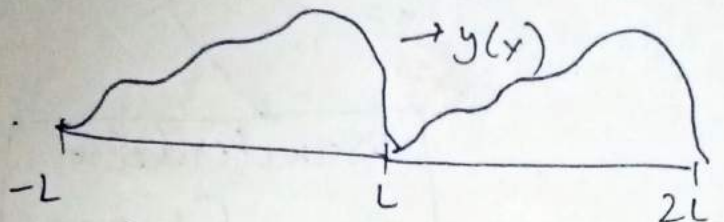
$$\Rightarrow \quad y(x) = \sum_{n=1}^{\infty}\frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

What does this mean? $\rightarrow$ Usual term by term plotting of the Fourier terms to show that it fits the ~~curve~~ pluck more and more.

when you pluck a string, you hear sounds made by normal mode vibrations — higher modes die down faster (will study later) (damped oscillation) $\rightarrow$ mishigh
Later

So, when we want a pure tone, we pluck and wait as the higher mode comp of the pluck fourier decomp will die down, and a few fundamentals die.

(III) General Fourier Series problem $\rightarrow$



$\rightarrow$ Repeated

We can write $y(x)$ as,

$$y(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \cos\left(\frac{m\pi x}{L}\right)$$

where,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} y(x)\,dx \quad , \quad a_n = \frac{1}{L}\int_{-L}^{L} y(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} y(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$\hookrightarrow$ Commit to memory (*)

1st Sept 2023

Wave equation $\rightarrow$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

works for any extended medium having wave (*)

① Standing wave solution $\rightarrow$ $\boxed{\begin{array}{l}\text{Boundary conditions} \\ y(0) = 0, \; y(L) = 0\end{array}}$

$\hookrightarrow$ These yield standing wave soln.

② Travelling wave solution.

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

□ Verify

$$\rightarrow \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) y = 0$$

(By multiplying out and using fact that order of partial derivatives do not matter, you can verify this)

Let us say y is a function of $(x - vt)$. → Guess ⊛ Crucial

$(x-vt)$ has nice translation property

Say, $z = x - vt \Rightarrow \frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial t} = -v$

y is now a func of z

$$\therefore \frac{\partial y}{\partial t} = \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial t} \Rightarrow \frac{dy}{dz}(-v) = \frac{\partial y}{\partial t}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{dy}{dz}\cdot(1)$$

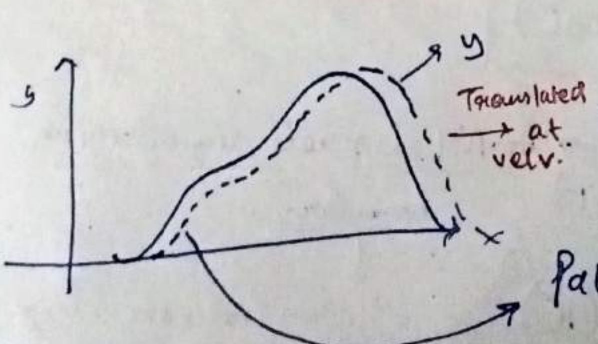
⊛ $x - vt$ is an educated guess because the variables are reversed w.r.t $\left(\frac{\partial y}{\partial t} + v\frac{\partial}{\partial x}\right)$ and reverse sign.

(Just work.) → This is a way of guessing solutions to these PDEs, if I had to guess.

Now, $\frac{\partial y}{\partial t} - v\frac{\partial y}{\partial x} = 0$

⊛ Separation is not required, but splitting makes it easier to arrive at solution.. (?) □ more PDE study required.

as, for $\frac{\partial}{\partial t} (\pm) v\frac{\partial}{\partial x}$, $y(x \mp vt)$ works.



$y(x - vt)$

What at, $y(x - v(t + \delta t))$?

Pattern just shifts by $v\delta t$.

⊛ v is velocity of the wave.

Our solution has no prediction on shape of wave, any shape works → Note that no matter what the shape is, it just must be a function of x−vt

⊛ Flipping sign on v ~~also~~ reverses direction too.
↳ −ve means prop in +ve x

Our standing wave solution was of the form,

$$\boxed{y = c \sin kx \cos \omega t}$$

$$\Rightarrow y = \frac{c}{2}\left[\sin(kx+\omega t) + \sin(kx-\omega t)\right]$$

$$\Rightarrow y = \underline{\frac{c}{2}\sin(kx+\omega t)} + \underline{\frac{c}{2}\sin(kx-\omega t)}$$

Two moving waves.

We had,

$$\boxed{\frac{\omega}{k} = v} \rightarrow \text{Again, IMP}$$

$$\Rightarrow y = \frac{c}{2}\sin\left\{k\left(x+\frac{\omega}{k}t\right)\right\} + \frac{c}{2}\sin\left\{k\left(x-\frac{\omega}{k}t\right)\right\}$$

$$\Rightarrow y = \frac{c}{2}\sin\{k(x+vt)\} + \frac{c}{2}\sin\{k(x-vt)\}$$

Left moving — Right moving

Standing wave is nothing but a superimposition of waves travelling to left and to right.

Can we then express any travelling wave solution as a sum of sines? We can see from a wave that each particle has only transverse movement — all with the same freq. — but phase shifted.

So, we guess a travelling wave solution,

$$\boxed{y = c\sin(kx - \omega t)}$$

⇒ In spherical, → Not discrete anymore as no boundary.

$$\therefore y = \int A e^{i(kx-\omega t)}$$

↳ $A \equiv f(k)$

IMP → Modes are no longer discrete

Amplitude is a function of K

Now,

$$y = \int A(K)\, e^{i(\omega Kx - \omega t)}\, dK$$

Why? we earlier used

$$y = \sum_m C_m \sin\left(\frac{m\pi}{L}x\right)\cos(\omega_m t + \phi)$$

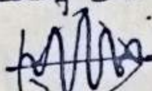
↓

Discrete sum as only discrete values allowed.

As it is now continuous, we integrate instead of discrete sum → Understand how this is converted to integration.

(*) Was distracted and missed what he said about generalisation of the solution. Get notes (*) Fill this in

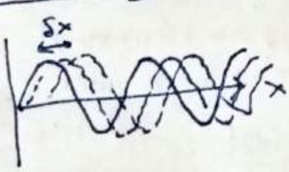
[+] To calculate Group vel, instead of discrete adds, we integrate → To get



→ moves with $\frac{d\omega}{dK}$.

$$\int C(K)\sin(Kx - \omega(K)t)\, dK$$

For travelling waves → Shift δx takes place in $\delta t \Rightarrow v = \frac{\partial x}{\partial t}$



Since the particles in the same phase move by the same $\delta x \Rightarrow$ Velocity is phase velocity (*)

Slowly moving waves → Superposition of two travelling wave solns.

$$\sin(K_1 x - \omega_1 t) + \sin(K_2 x - \omega_2 t)$$

$$\Rightarrow 2\sin\left(\frac{K_1+K_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{K_1-K_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$$

If K_1 and K_2 are close, K_1+K_2 has small λ → oscillates fast, K_1-K_2 has big λ → oscillates slow.

Smaller travelling wave (K is larger → λ is smaller)

⇓

$$\boxed{\lambda = \frac{4\pi}{K_1+K_2}}$$ (*)

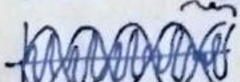
Larger λ as K_{eff} is smaller

⇓

$$\boxed{\lambda = \frac{4\pi}{K_1-K_2}}$$ (*)

→ What matters is how fast the envelope moves

When K_1 and K_2 are very close → $\frac{\omega_1-\omega_2}{K_1-K_2} = \frac{\Delta\omega}{\Delta K} = \boxed{\frac{d\omega}{dK}}$ (*)



→ wave packet

↳ moves with velocity $\boxed{\frac{d\omega}{dK}}$ → Group velocity (*)

↓

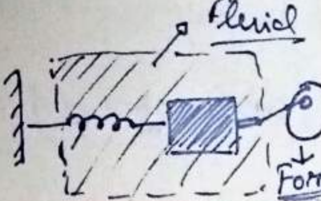
Slower (envelope)

[+] Continued

5th September 20..

Forced Damped SHO →

To understand refraction and reflection of waves, we need to understand ~~forced SHO~~. forced-damped SHO



How do we write eqn if this is in some fluid and there is drag?

Force viscous force → Because it needs.

$$\boxed{m\ddot{x} = -Kx - \gamma\dot{x} + F_0 \cos\omega t}$$ ⊛ IMP

↳ Forcing.

Why are there viscous forces / drag force?

Static fluid.

→ Fluid attached to surface due to cohesion.

Friction b/w layers — viscous force. → Einstein's relation and shit

So, the eqn is,

⊛ Note that other velocity dependent forces also work

☐ What about non-vel dep forces? (Nonlinear?)

$$m\ddot{x} = -Kx - \gamma\dot{x} + F_0\cos\omega t$$

$$\Rightarrow m\ddot{x} + \gamma\dot{x} + Kx = F_0\cos\omega t$$

ω : forcing freq

⊛ If the equation has to hold for all time, x needs to have a freq of ω → very qualitative leap of logic there. ☐ Better way?

~~Hom~~ Non-homogeneous 2nd order ODE.

$\Rightarrow y_h$ and y_p are solution,

$y_h \downarrow x_2$. $y_p \downarrow x_1$

Linearity / Superposition

$$m\frac{d^2}{dt^2}(x_1 + x_2) + \gamma\frac{d}{dt}(x_1 + x_2) + K(x_1 + x_2) = F_0\cos\omega t$$

x_1 : Particular integral

x_2 : Complementary solution.

⊛ $x_1 + c x_2$ is a solution to our ODE.

We will separately solve them, and find physical meaning.

$$\boxed{\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0}$$ (complementary) → homogeneous.

Notation → $\boxed{\frac{\gamma}{m} = \alpha}$, $\boxed{\frac{k}{m} = \omega_0^2}$ → ⊛

↳ usual, though

$$\Rightarrow \ddot{x} + \alpha + \omega_0^2 x = 0$$

→ usual solution guess.

Tayamatz → $x = Ae^{i\omega_2 t}$

$$\dot{x} = Ai\omega_2 e^{i\omega_2 t}, \quad \therefore \ddot{x} = -A\omega_2^2 e^{i\omega_2 t}$$

$$\therefore \left(-\omega_2^2 + \alpha i\omega_2 + \omega_0^2\right) = 0$$

$$\Rightarrow \omega_2 = \frac{i\alpha}{2} \pm \frac{1}{2}\sqrt{-\alpha^2 + 4\omega^2}$$

$$\Rightarrow \omega_2 = \frac{i\alpha}{2} \pm \omega_1, \quad \omega_1 = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$$

$$\therefore x_2 = Ae^{i\left(\frac{i\alpha}{2} + \omega_1\right)t} + Be^{+i\left(\frac{i\alpha}{2} - \omega_1\right)t}$$

$$\Rightarrow x_2 = e^{-\frac{\alpha}{2}t}\left[Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right]$$

→ Real part

$$= e^{-(\alpha/2)t}\, C\cos(\omega_1 t + \phi)$$

Initial conditions (Sam) →

$$x(0) = a$$

$$\dot{x}(0) = 0$$ → zero initial velocity.

$$\therefore x(0) = \cancel{\cos}\, C\cos\phi = a \quad \text{— (I)}$$

$$\therefore \dot{x} = -\frac{\alpha}{2}Ce^{-\frac{\alpha}{2}t}\cos(\omega_1 t + \phi) - \omega_1 C e^{-\frac{\alpha}{2}t}\sin(\omega_1 t + \phi)$$

$$\Rightarrow \dot{x}(0) = -\frac{\alpha}{2}C\cos\phi - \omega_1 C\sin\phi = 0 \quad \text{— (II)}$$

$\Rightarrow C\sin\phi = -\frac{\alpha}{2}\frac{a}{\omega_1}$

$C\cos\phi = a$

~~$\therefore x_2(t) = e^{-\frac{\alpha}{2}t} C[\cos\omega_1 t \cos\phi - \sin(\omega_1 t)$~~ NOW. Ans. Wondart.

$x_2(t) = e^{-\frac{\alpha}{2}t} C\left[\cos\omega_1 t \cos\phi - \sin(\omega_1 t)\sin\phi\right]$

$= e^{-\frac{\alpha}{2}t}\left[a\cos\omega_1 t + \frac{\alpha}{2}\frac{a}{\omega_1}\sin\omega_1 t\right]$ (*)

Plot this → (*) $\alpha < \omega$, [Damping much smaller]

Imp assumption.

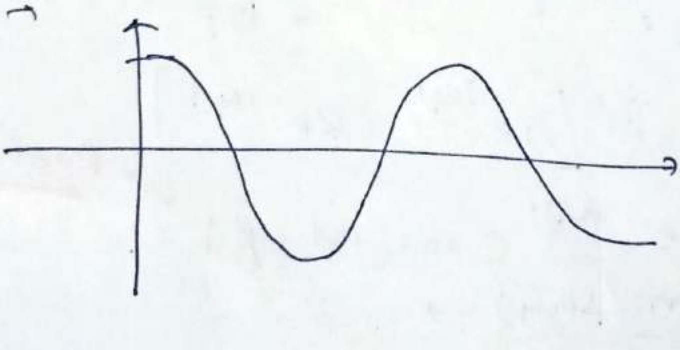
ω_0 freq changes to $\boxed{\omega_1 = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}}$ (*), but in practice, $\omega_p \approx \omega_0$

(Underdamped case) → α is small, often

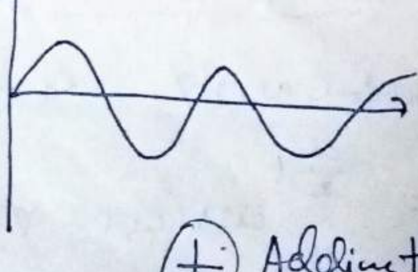
$\therefore \frac{\alpha a}{2\omega_1}\sin\omega_1 t$ → Small amplitude.

Term by term →

$a\cos\omega_1 t$ →

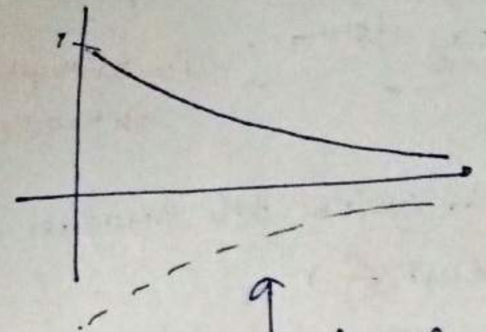


$\frac{\alpha a}{2\omega_1}\sin\omega_1 t$ →



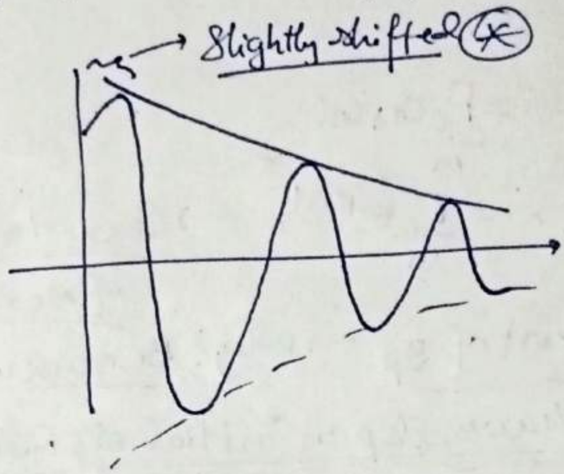
(+) Adding them, we get phase shifted sinusoid.

$e^{-\frac{\alpha}{2}t}$ →



When you have a ~~slow~~ slow function and a fast function (cos)

then, fit fast function inside ~~fast~~ slow function envelope.



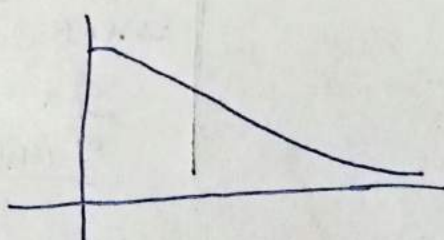
Usual envelope behavior

!! Notes not very thorough in this part — read book

(*) We can not often measure the shift.
↳ of ω from ω_0

Other situations →

What if $\frac{\alpha^2}{4} = \omega_0^2 \Rightarrow \omega_1 = 0$ (Critically damped)



→ $x(t)$.

→ Goes to zero almost immediately.

(*) Imp for ~~se~~ sensitive equipment.

Now, we take $\lim_{\omega_1 \to 0} \left(\frac{\alpha a}{2\omega_1} \sin \omega_1 t \right) \approx Bt$

∴ $\boxed{x(t) = e^{-\frac{\alpha}{2}t}(A + Bt)}$ → Reaches zero faster than overdamp.

Linear part

On overdamp, all the ~~f~~ trig functions are replaced by hyperbolic trig functions — all decay term. → Due to complex terms being introduced.

Forcing next term.

☐ Try solving spring-mass system with surface friction, i.e, non vel dependent

---

8th Sept '2023

We start with particular solution for the forced damped SHO diff eqn.

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

$$\Rightarrow \ddot{x} + \alpha\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \rightarrow \text{Conv. to complex no. for ease.}$$

We take the real part of y_p → Why? Because we will see that y_p does not have dep on initial ~~c~~ conditions. → So we need to take real part, as no initial conditions can make it real.

↳ Again, quite the leap in logic here.

We try guess,

$$x = Ae^{i\omega t}$$

$$\Rightarrow \dot{x} = Ai\omega e^{i\omega t}$$

$$\Rightarrow \ddot{x} = -A\omega^2 e^{i\omega t}$$

Why? We are looking for steady state oscillation essentially at large value of t → So no effect of initial condition.

$$\therefore -\omega^2 Ae^{i\omega t} + \alpha i\omega Ae^{i\omega t} + \omega_0^2 Ae^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

If this is to hold for all time,

$$A(-\omega^2 + \alpha i\omega + \omega_0^2) = \frac{F_0}{m}$$

$$\Rightarrow \boxed{A = \frac{F_0}{m} \cdot \frac{1}{i\alpha\omega + (\omega_0^2 - \omega^2)}}$$ ⊛ IMP

↳ A is completely known.

⊛ → So what about initial condition? The complementary solution takes care of that → This is why the linear combs are important.

Useful to understand Concept of resonance

↓

⊛ Crucial: If we have a system that has natural freq ω_0 and we force it at ω_0, then the system will oscillate with large amplitudes.

⊛ → If we have,

$$a+ib = \sqrt{a^2+b^2}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos\theta} + i\,\underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin\theta}\right)$$

$$= \underbrace{\sqrt{a^2+b^2}}_{\text{Scalar}} \cdot e^{i\theta}, \quad \tan\theta = \frac{b}{a}$$

We do this with expression for A, to get real part.

$$\therefore A = \frac{F_0}{m} \frac{1}{i\alpha\omega + (\omega_0^2 - \omega^2)}$$

mult ~~div~~ up and down with i

$$= \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cdot e^{-i\phi}$$

$$\phi = \tan^{-1}\left(\frac{\alpha\omega}{\omega_0^2 - \omega^2}\right)$$

⊛ Note → The independence of y_p from initial conditions has large consequences.

☐ Read more. ← Like what ?

$$\Rightarrow A = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cdot e^{-i\phi}$$

~~A~~ An, $x = Ae^{i\omega t}$

$$\Rightarrow x = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cdot e^{i(\omega t - \phi)}$$

Real part

$$\Rightarrow \boxed{x = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cos(\omega t - \phi)}$$

y_p

IMP

We plot the expression for A →

$$A = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}}.$$

At $\omega = 0$, $A = \frac{F_0}{m} \cdot \frac{1}{\omega_0^2}$ → No drive

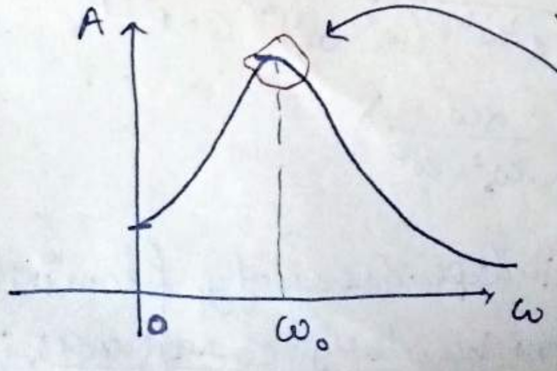
At $\omega = \omega_0$, $A = \frac{F_0}{m} \cdot \frac{1}{\alpha\omega_0}$ → If we stick to the small α (underdamp) condition,

↓ Resonance

$$\frac{F_0}{m} \cdot \frac{1}{\omega_0^2} < \frac{F_0}{m} \frac{1}{\alpha\omega_0}$$

At $\omega \gg \omega_0$, $A \approx \frac{F_0}{m} \frac{1}{\omega^2}$ → falls off inverse square.

↓ over driven



Resonance phenomenon

I ~~Calculation of power~~ → :

(*) Demonstration of pohl's pendulum next week.

If we have a motor that is forcing the SHO, ~~and~~ if we monitor the current consumed by it as a function of forcing freq ω, we will see that it follows the $\boxed{\text{A vs. } \omega \text{ curve}}$ → Same curve

Why is this important? ~~I~~ Since this happens in all SHO → we can study even quantum systems by just reading off of ammeters.

$$dW = F_0 \cos\omega t \, dx \quad \rightarrow \text{Work done by forcing.}$$

$$\Rightarrow W = \int_0^T F_0 \cos\omega t \cdot \frac{dx}{dt} \cdot dt \quad , \quad T = \frac{2\pi}{\omega}$$

$$= \int_0^T F_0 \cos\omega t \cdot \dot{x} \, dt \quad \rightarrow \text{Over one entire cycle.}$$

Why?
Current is AC, as the force is oscillating and not fixed.

Average over one cycle is useful. ←

~~Power = ⟨P⟩~~ → Average power ⟨P⟩

$$\Rightarrow \langle P \rangle = \frac{W}{T} = \frac{1}{T} \int_0^T F_0 \cos\omega t \cdot \dot{x} \, dt .$$

Like, $\langle x \rangle = \int_0^T v \, dt = 0$ → oscillator is mean at 0.

$$x = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}} \left[\cos\omega t \cos\phi + \sin\omega t \sin\phi\right]$$

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}} \; ; \; \sin\phi = \frac{\alpha\omega}{\sqrt{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}}$$

$$\Rightarrow \cancel{x = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}}}$$

$$\rightarrow x = \frac{F_0}{m\{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2\}} \cdot \left[(\omega_0^2-\omega^2)\cos\omega t + \alpha\omega \sin\omega t\right]$$

$$\Rightarrow \boxed{x = A_{dis}\cos\omega t + A_{ab}\sin\omega t}$$

$$A_{dis} = \frac{F_0}{m} \cdot \frac{\omega_0^2-\omega^2}{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}$$

↓

Dispersive amplitude.

$\frac{\pi}{2}$ phase shifted from ω.

$$A_{ab} = \frac{F_0}{m} \frac{\alpha\omega}{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}$$

↓

Why? ← Absorptive amplitude.

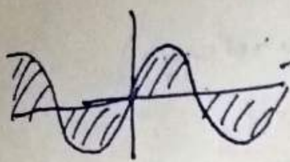
↓

The amplitude that has a $\frac{\pi}{2}$ phase shift from forcing has an absorptive power → Absorbs energy from forcing.

$$\therefore \dot{x} = -\omega A_{dis} \sin \omega t + \omega A_{ab} \cos \omega t$$

$$\therefore W = F_0 \int_0^T [\underbrace{-\omega A_{dis} \sin \omega t \cos \omega t}_{-\frac{\omega A_{dis}}{2} \sin 2\omega t} + \underbrace{\omega A_{ab} \cos^2 \omega t}_{\frac{\omega A_{ab}}{2}(1+\cos 2\omega t)}]\, dt$$

$$\Rightarrow W = F_0 \int_0^T \left[-\frac{\omega A_{dis}}{2} \sin 2\omega t + \frac{\omega A_{ab}}{2} (1 + \cos 2\omega t) \right] dt$$



→ Area cancels out if we integrate over integral multiples of time period of sinusoid.

~~Only (1+~~ $(1 + \cos 2\omega t)$ → Only this survives integral.

$$\therefore \langle P \rangle = \frac{F_0^2}{2m\alpha} \left[\frac{\alpha^2 \omega^2}{\alpha^2 \omega^2 + (\omega_0^2 - \omega^2)^2} \right]$$

$$= \frac{F_0^2}{2m\alpha} \left[\frac{1}{1 + \frac{(\omega_0^2 - \omega^2)^2}{\omega^2 \alpha^2}} \right]$$

We try to plot $\langle P \rangle$

(*) Note: the wants us to be meticulous when asked to plot in exam.

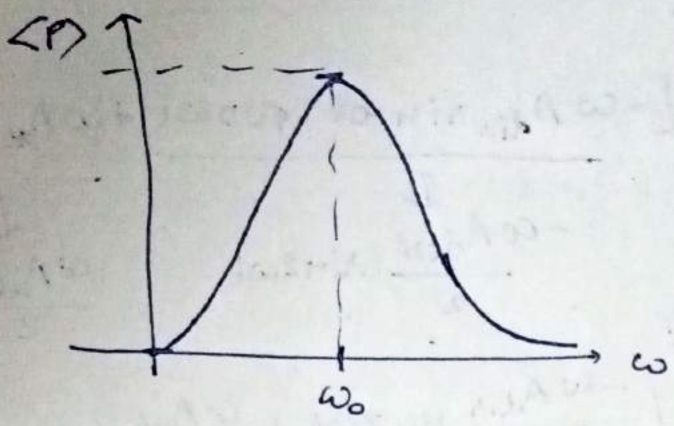
At $\omega = 0$, $\langle P \rangle = 0$

At $\omega = \omega_0$, $\langle P \rangle = \frac{F_0^2}{2m\alpha}$ ↑ Same as Avs. ω.

At $\omega >> \omega_0$, $\langle P \rangle = \frac{F_0^2}{2m\alpha} \cdot \frac{1}{1 + \frac{\omega^2}{\alpha^2}}$, decays inverse square.

On diff $\langle P \rangle$ w.r.t ω, we find extrema at $\omega = \pm \omega_0$ → we find $\langle P \rangle(\omega_0)$ to see if it is max or min.

Plot →



(*) Roughly symmetric → Almost exact symmetry for small damping (underdamp) (unlike amplitudes) → ☐ Verify

How do we define the sharpness of this <P> vs. ω peak?

(*) Define.

(*) Quality factor → $\boxed{Q = \frac{\omega_0}{\alpha}}$ → Directly proportional to 'sharpness'.

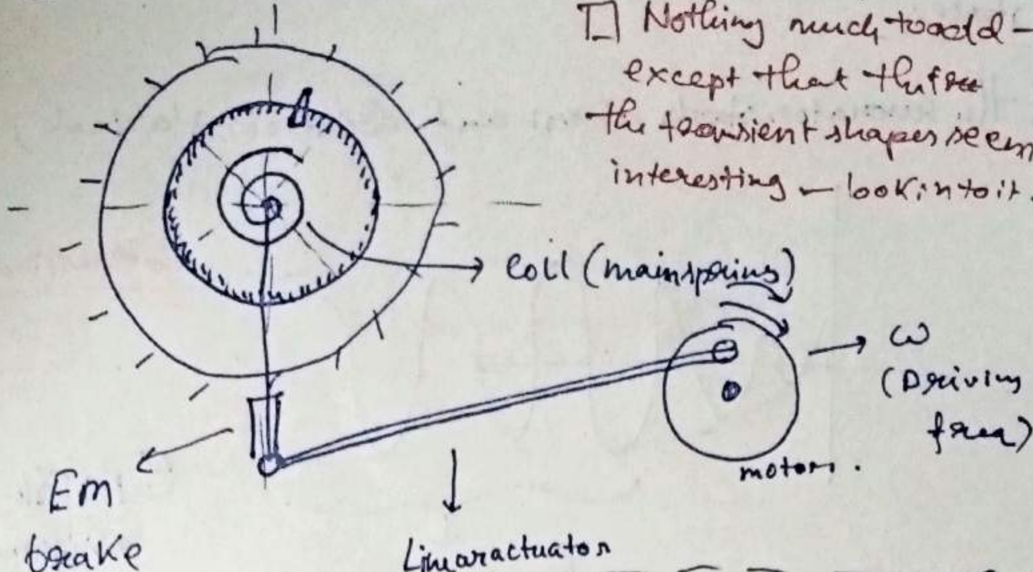
☐ Check motivation for definition

(*) Class Test next week.

12th sept 2023

Demonstration class →

[I] Nothing much to add – except that the ~~free~~ the transient shapes seem interesting – look into it.



$(I \propto \alpha)$

$$\boxed{\ddot{x} + \alpha\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t}$$

↳ current in em brake $\propto \alpha$ (damp coeff)

→ You set current through ~~...~~ EM to low (underdamp)

→ You set ω to max, then decrease ~~...~~ until resonance is ~~...~~ reached.

(*) Damping coeff may be altered to show that $A_{res} \propto \frac{1}{\alpha}$

(*) (*) Damping can be turnt off to show very large amplitude at resonance.

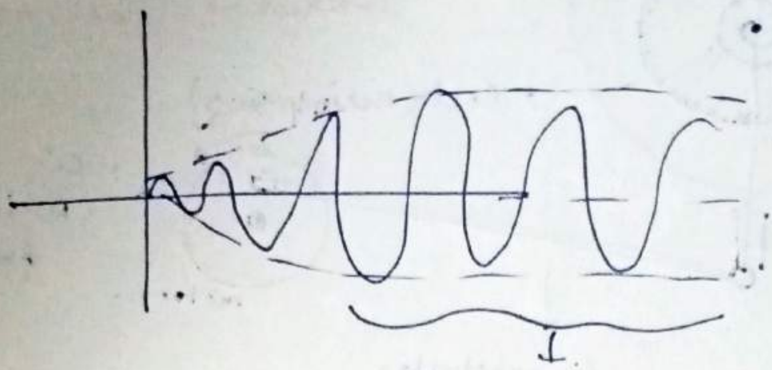
Note: There is a transient state between changes that needs to be given time to die down.

If we plot the amplitudes, we find the usual A vs. ω curve, if we find power via current, we obtain P vs. ω curve.

(*) Very near resonance, $\omega_0 \approx \omega$, we get the motion to be a linear comb of y_p and y_h as usual – the $\omega_0 - \omega$ and $\omega_0 + \omega$ causes a beat phenomena.

(*) We can stop the driver b/w states to minimise transient states.

If the resonator starts at rest and driver is started,



Transients die when A is constant.

Interesting

(*) Possible assignment ques..

Conch shell → (*) paper on acoustics

6 cm → 455 Hz

(6cm in the length from mouth where diameter is max).

7 cm → 423.75 Hz.

7.5 cm → 420 Hz.

10.5 cm → 307 Hz.

→ Should be roughly linear if plotted

D Try.

In the assignment problem with checking Lorentz invariance

→ Do it with Galilean transformations too.

This way is instructive,

$$\therefore \frac{\partial^2 y}{\partial t^2} = v_0^2 \frac{\partial^2 y}{\partial x^2}$$

O.K.

We can re-write as,

$$\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x}\right) y = 0$$

whew, over.

Good job

Observe change here.

$$\frac{\partial^2 y}{\partial t^2} = v_0^2 \frac{\partial^2 y}{\partial x^2}$$

O.K.

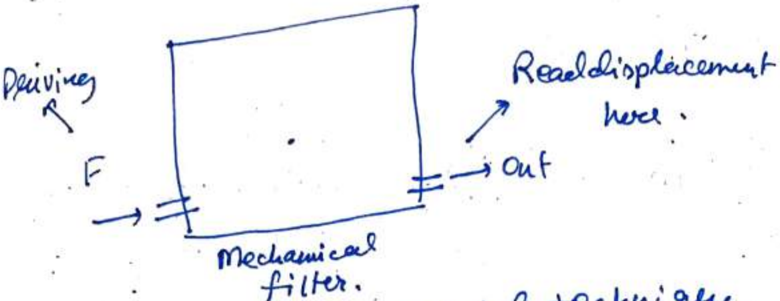
We can re-write as,

$$\left(\frac{\partial}{\partial t} - v_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)y = 0$$

Observe change here.

whew, over.
Good job

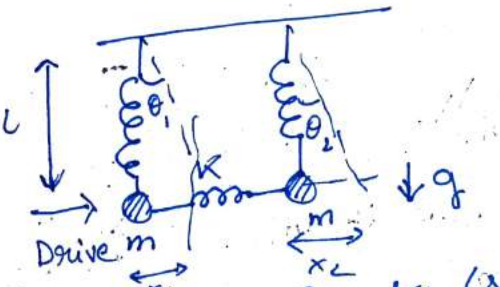
15th Sept 2023



(*) Concept today is imp experimental technique.

We look at coupled pendulum.

~~$mL^2\ddot{\theta}_1 = mgL\sin\theta_1$~~



$$mL^2\ddot{\theta}_1 = -mgL\sin\theta_1 - KL(\sin\theta_1 - \sin\theta_2)\cdot L\cos\theta_1$$

$$\Rightarrow \ddot{\theta}_1 = -\frac{g}{L}\theta_1 - \frac{K}{m}(\theta_1 - \theta_2)$$

$$\Rightarrow \boxed{\ddot{\theta}_1 = -\left(\frac{g}{L} + \frac{K}{m}\right)\theta_1 + \frac{K}{m}\theta_2}$$

We ~~cal~~ can also this to be transverse displacement under small angle approx

$$L\theta_1 \sim x_1, \quad L\theta_2 \sim x_2$$

$$\Rightarrow \ddot{x}_1 = -\left(\frac{g}{L} + \frac{K}{m}\right)x_1 + \frac{K}{m}x_2, \quad \ddot{x}_2 = -\left(\frac{g}{L} + \frac{K}{m}\right)x_2 + \frac{K}{m}x_1$$

Equations of damping.

Assuming, for normal modes,

$$\ddot{X} = \ddot{x}_1 + \ddot{x}_2$$

$$\Rightarrow \boxed{\ddot{X} = -\frac{g}{L}X}$$

Decoupled.

$$\ddot{Y} = \ddot{x}_1 - \ddot{x}_2$$

$$\Rightarrow \boxed{\ddot{Y} = -\left(\frac{g}{L} + \frac{2K}{m}\right)Y}$$

We now connect damping to left pendulum, and add some damping,

$$\ddot{x}_1 = -\left(\frac{g}{L} + \frac{K}{m}\right)x_1 + \frac{K}{m}x_2 \underbrace{- \frac{\gamma}{m}\dot{x}_1}_{\alpha,\ \text{damp}} \underbrace{+ \frac{F_0}{m}\cos\omega t}_{\text{forcing}}$$

$$\Rightarrow \boxed{\ddot{x}_1 + \alpha\dot{x} + \left(\frac{g}{L} + \frac{K}{m}\right)x_1 - \left(\frac{K}{m}\right)x_2 = \frac{F_0}{m}\cos\omega t}$$

Similarly,

$$\boxed{\ddot{x}_2 + \alpha\dot{x}_2 + \left(\frac{g}{L} + \frac{K}{m}\right)x_2 - \left(\frac{K}{m}\right)x_1 = 0}$$

☐ What happens if they have different damping for each mass?

Now, they are not symmetric

We find modes (notice that works still)

$$\ddot{X} + \alpha\dot{X} + \frac{g}{L}X = \frac{F_0}{m}\cos\omega t$$

$$\ddot{Y} + \alpha\dot{Y} + \left(\frac{g}{L} + \frac{2K}{m}\right)Y = \frac{F_0}{m}\cos\omega t$$

} Both got damp and forcing.

Now, let us find steady state soln. (y_p, basically)

$$\therefore \boxed{X_{SS} = \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_1^2-\omega^2)^2}} \cos(\omega t + \phi_1)}$$

and,

$$\boxed{Y_{SS} = \frac{F_0}{m} \cdot \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_2^2-\omega^2)^2}} \cdot \cos(\omega t - \phi_1)}$$

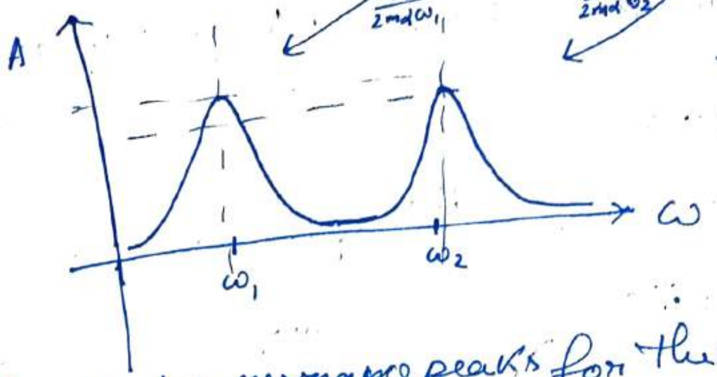
They act like single oscillators in each mode.

Say we are interested in individual solutions →

$$\cancel{x_{2_{SS}} = \frac{1}{2}} \quad x_2^{SS} = \frac{1}{2}(X_{SS} - Y_{SS})$$

$$\Rightarrow x_2^{SS} = \frac{1}{2} \cdot \frac{F_0}{m} \left[\underbrace{\frac{\cos(\omega t + \phi_1)}{\sqrt{\alpha^2\omega^2 + (\omega_1^2-\omega^2)^2}}}_{\frac{F_0}{2m\alpha\omega_1}} - \underbrace{\frac{\cos(\omega t - \phi_2)}{\sqrt{\alpha^2\omega^2 + (\omega_2^2-\omega^2)^2}}}_{\frac{F_0}{2m\alpha\omega_2}} \right]$$

Plot →



Two peaks for two resonance values.

So, there are two resonance peaks for the amplitude of x_2^{SS}.

(*) We can forget about the cosine effect here as we can just combine

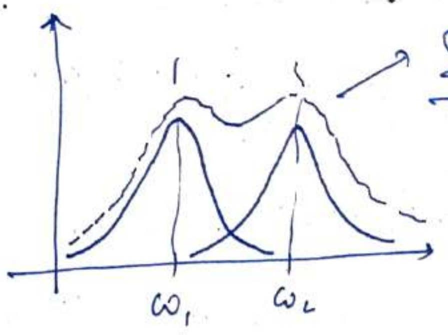
$$\frac{1}{\omega_1} \cos(\omega t - \phi_1) \; ~~and~~ \; - \frac{1}{\omega_2} \cos(\omega t - \phi_2)$$

into one sinusoid.

Note that this generalises nicely — if there are n-oscillators that are coupled — we can find n resonance peaks if we drive over a range of frequencies.

(*) In general, the expression of motion of any one oscillator in a coupled n-oscillator can be expressed as linear comb. of n resonance terms ~~reson~~ oscillating at diff. phase.

Now, back to the mechanical filter. Say we drive over a range of freq.



Sum → This is what we read as motion amp of oscillation

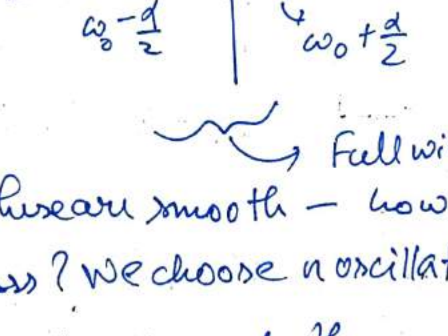
(What made the peaks far before? $\frac{k}{m}$ was appreciab.)

Note that only these freq b/w ω_1 and ω_2 cause oscillations (at almost same amp)

This is a band pass filter → Allows only ~~one~~ some freq to pass.

(*) Imp in condensed matter, optics, etc..

Single oscil'n



d = damp

→ Power drops from max to $\frac{1}{2}$ value.
(Amp drops by $\frac{1}{\sqrt{2}}$ factor)

→ Full width at half max

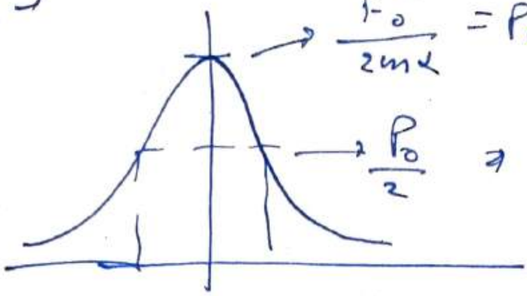
But these are smooth – how do we make it a good bandpass? We choose n oscillators with sharp peaks (resonance) near each other.

→ Becomes sharper

$$P(\omega) = \frac{F_0^2}{2m\alpha}\left[\frac{\alpha^2\omega^2}{\alpha^2\omega^2 + (\omega_0^2-\omega^2)^2}\right]$$

$$= \frac{F_0^2}{2m\alpha}\left[\frac{1}{1 + \frac{(\omega_0^2-\omega^2)^2}{\alpha^2\omega^2}}\right]$$

<u>Plot</u> →



$\rightarrow \frac{F_0^2}{2m\alpha} = P_0$

$\rightarrow \frac{P_0}{2} \Rightarrow 1 + \frac{(\omega_0^2-\omega^2)^2}{\alpha^2\omega^2} = 2$

$$\Rightarrow \frac{(\omega_0^2 - \omega^2)^2}{\alpha^2\omega^2} = 1 \Rightarrow \omega_0^2 - \omega^2 = \pm\alpha\omega$$

$$\Rightarrow \omega^2 \pm \alpha\omega - \omega_0^2 = 0$$

We ignore ~~+ve~~ -ve ω for ease here (same result)

$$\Rightarrow \omega^2 + \alpha\omega - \omega_0^2 = 0.$$

$$\therefore \omega = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 + 4\omega_0^2}$$

$$\Rightarrow \omega = -\frac{\alpha}{2} \pm \sqrt{\omega_0^2 + \frac{\alpha^2}{4}}$$ → very small. (ignore)

$$\Rightarrow \omega = \omega_0 - \frac{\alpha}{2} \text{ and } -\left(\omega_0 + \frac{\alpha}{2}\right)$$

(*) Note that the diff b/w these roots is $\underline{\underline{\alpha}}$.

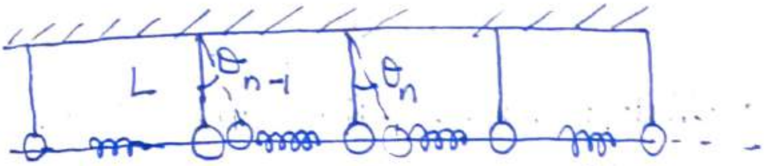
α → width of resonance at <u>half width</u>

Next we will extend filter to ~~n-os~~ n-coupled oscillators

→ we that to understand reflection of wave.

<u>(TIR, etc)</u>

19/09/2023



Ang. acc

$$mL^2 \ddot{\theta}_n = -mgL\sin\theta_n - k_s L(\sin\theta_n - \sin\theta_{n+1})L\cos\theta_n$$
$$- k_s L(\sin\theta_n - \sin\theta_{n-1})L\cos\theta_n$$

→ small ∠ app.

$$\boxed{\ddot{\theta}_n = -\frac{g}{L}\theta_n - \frac{k_s}{m}(2\theta_n - \theta_{n-1} - \theta_{n+1})}$$

→ Beaded string exp.

$$L\theta_n = y_n$$
$$\ddot{y}_n = -\frac{g}{L}y_n - \frac{k_s}{m}(2y_n - y_{n-1} - y_{n+1})$$

Taking continuum limit,

$$\ddot{y}_n = -\frac{g}{L}y_n + \frac{k_s}{m}a^2\left(\frac{\frac{y_{n+1}-y_n}{a} - \frac{y_n - y_{n-1}}{a}}{a}\right)$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = -\frac{g}{L}y + \frac{k_s a^2}{m}\cdot\frac{\partial^2 y}{\partial x^2}}$$

Cline? Border? Gordon?
Some ~~Bhodeh~~ eq.

Cline - Gordon Equation

↳ Looks like it, that is.

How do we find out how the normal mode freq.s (ω) relate to k?

We assume, all oscillators move with same ω and find out how it relates to k.

So, we assume, $y_n = A_n \cos(\omega t + \phi)$.

$$y_{n-1} = A_{n-1} \cos(\omega t + \phi), \; y_{n+1} = A_{n+1} \cos(\omega t + \phi)$$

$$\therefore \; -\omega^2 A_n \cos(\omega t + \phi) = -g/L \, A_n \cos(\omega t + \phi) - \frac{k_s}{m}\left(2A_n \cos(\omega t + \phi) - (A_{n+1} + A_{n-1})\cos(\omega t + \phi)\right)$$

$$\Rightarrow -\omega^2 A_n = -g/L \, A_n - \frac{k_s}{m}\left(2A_n - A_{n+1} - A_{n-1}\right)$$

$$\Rightarrow -\omega^2 A_n = -\omega_0^2 A_n - \frac{k_s}{m}\left(2A_n - A_{n+1} - A_{n-1}\right)$$

We observe, that,

$$\frac{k_s}{m}\left(2A_n - A_{n+1} - A_{n-1}\right) = \left(\omega^2 - \omega_0^2\right) A_n$$

$$\Rightarrow \boxed{\frac{\left(2A_n - A_{n+1} - A_{n-1}\right)}{A_n} = \frac{\omega^2 - \omega_0^2}{k_s/m}}$$

LHS as a func. of n, RHS not a func. of n.

↳ So we choose A_n s.t. LHS is independent of n

$$\boxed{A_n = C \sin n\theta}$$

$$\Rightarrow A_{n-1} + A_{n+1} = C\sin(n-1)\theta + C\sin(n+1)\theta$$

$$= C\left[\sin\frac{(n-1+n+1)\theta}{2}\cos\frac{(n-1-n-1)\theta}{2}\right]$$

$$= 2C\left[\sin n\theta\cos\theta\right]$$

Substituting in LHS

$$\frac{\left[2C\sin n\theta - 2C[\sin n\theta\cos\theta]\right]}{C\sin n\theta} = 2(1-\cos\theta)$$

$$= \frac{\omega^2-\omega_0^2}{K_s/m}$$

$$\therefore 2.2\sin^2\theta/2 = \frac{\omega^2-\omega_0^2}{K_s/m}$$

$$\Rightarrow \boxed{\omega^2 = \omega_0^2 + 4K_s/m\,\sin^2\theta/2}$$

→ We could have got this from C.L as well, more elegantly.

This tells us, for CL $\boxed{f(x) = \sin kx}$ & discrete $\boxed{f(n) = \sin n\theta}$

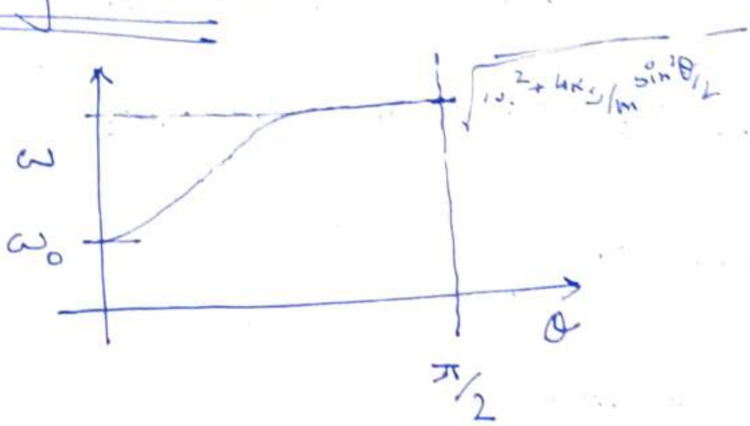
y = some $\left(f(x) = A\right)\cos(\omega t+\phi)$

Eq. takes form $\boxed{-\omega^2 f = -\omega_0^2 f + \frac{K_s}{m}a^2\frac{\partial^2 f}{\partial x^2}}$

Eq. for harmonic oscillator.

$$\frac{d^2f}{\partial x^2} = -\frac{(\omega^2-\omega_0^2)f}{K_s a^2/m} \qquad \boxed{f = C\cos(Kx+\phi)}$$

Plotting ω vs. θ.



$\theta = 0$ $\quad \omega = \omega_0$ [in-phase sol. for 2-pendula]

For beaded string we calculated θ from boundary condition, $\theta = ka$,

but here no boundary.

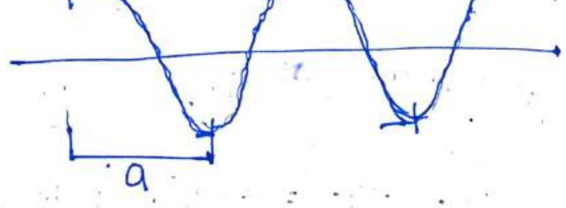
Starting with sine (can be sine/cos/comb)

Here also, $\theta = ka$.

If we say, this is the highest energy possible.

All the next pendula are relatively out of phase, so what is the periodicity?

(highest freq) ω_{max}



$\sin\theta/2 = 1$

$\theta = \pi$

$\downarrow$

$ka = \pi$

ω_{max} corresponds to out of phase, so alt. beads ~~are~~ must be in-phase

$\Rightarrow \frac{2\pi}{\lambda} a = \pi$

$\Rightarrow \boxed{a = \frac{\lambda}{2}}$ $\quad \boxed{\lambda = 2a}$

If we plug ka in the eq, we get relation between ω & k

The upper limit for ω is due to the sine x component.

↳ always the case for physical situations (say even rows of atoms are not continuous)

⊛⊛ Comparing CL & discrete,

$C \sin k x \rightarrow C \sin n\theta$

$\rightarrow \underline{kx \rightarrow} \; k_n a$

$$\sin^2 \theta/_2 = \frac{(\omega^2 - \omega_0^2)}{4k_s/m}$$

If $\omega < \omega_0 \rightarrow \theta = \text{imaginary}$

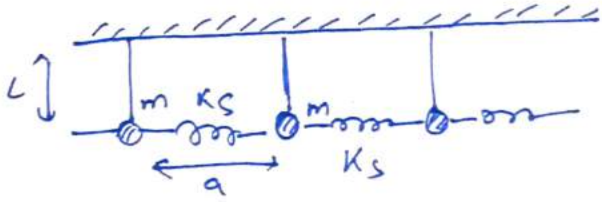
$\omega > \omega_0$,

$\omega^2 - \omega_0^2 < 4k_s/m \rightarrow \theta = \text{imaginary}$

$\underline{\sin \theta_{/2} > 1}$

⇒ $\boxed{\text{whenever the driving frequency is less than lower limit, or above upper limit; } \theta \rightarrow \text{imaginary}}$

22nd Sept 2023



n - coupled pendula.

$\overrightarrow{F_0 \cos \omega t}$ → forcing

(*) Once ~~all~~ all the transients have died or down, all the pendula will oscillate at ω. (Steady)

We wrote the discrete eqn as —

$$\boxed{\ddot{y}_n = -\frac{g}{L} y_n - \frac{K_S}{m}\left(2y_n - y_{n-1} - y_{n+1}\right)}$$

Using, $\dfrac{y_{n+1} - y_n}{a} \approx \left.\dfrac{\partial y}{\partial x}\right|_n$, we formulate continuous

$$\boxed{\frac{\partial^2 y}{\partial t^2} = -\frac{g}{L} y + \frac{K_S a^2}{m}\frac{\partial^2 y}{\partial x^2}}$$

Of the form of the Klein-Gordon wave eqn

(*) Discussion regarding how this eqn describes oscillations in the ionosphere. → Qualitative.

We attempt to solve the KG wave eqn for the steady state of this n-coupled oscillator.

Assume,

$$y(x,t) = f(x)\cos(\omega t + \phi)$$

Let, $\omega_0^2 = \dfrac{g}{L}$, $\omega_S^2 = \dfrac{K_S}{m}$

∴ Putting into eqn, we have,

$$-\omega^2 f = -\omega_0^2 f + \omega_s^2 a^2 \frac{d^2 f}{dx^2}$$

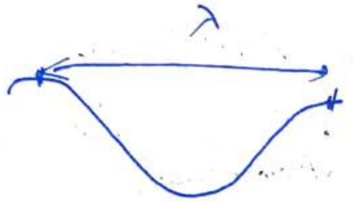
$$\Rightarrow \frac{d^2 f}{dx^2} = -\left(\frac{\omega^2 - \omega_0^2}{\omega_s^2 a^2}\right) f \quad \rightarrow \text{Reminds us of SHM.}$$

$$\Rightarrow \frac{d^2 f}{dx^2} = -K^2 f$$

We assume usual SHM periodic solution –

$$f(x) = C \cos Kx + D \sin Kx$$
$$= C \cos(Kx + \phi)$$

Now,



→ One complete cycle = 2π.

$$K(x+\lambda) - Kx = 2\pi \quad \Rightarrow K\lambda = 2\pi$$

$$\Rightarrow \boxed{K = \frac{2\pi}{\lambda}}$$

(*) → Not much more to be done here – but it gives us a clue as to what we may do in discrete case.

The discrete solution must then be of the form,

$$y_n = A_n \cos(\omega t + \phi)$$

↳ works same as $f(x)$ in continuous case.

$$\Rightarrow A_n = C \cos(Kx)\big|_n + D \sin(Kx)\big|_n$$
$$= C \cos(Kna) + D \sin(Kna)$$

Plugging this into discrete differential eqn,

$$-\omega^2 A_n = -\omega_0^2 A_n - \omega_s^2 (2A_n - A_{n-1} - A_{n+1})$$

$$\Rightarrow \omega^2 = \omega_0^2 + \omega_s^2\left(2 - \frac{A_{n+1} - A_{n-1}}{A_n}\right)$$

We rearrange,

$$\frac{A_{n+1}+A_{n-1}}{A_n} = 2\cos Ka \quad \text{(Same as usual, here we even know what } A_n \text{ is)}$$

$$\Rightarrow \boxed{\omega^2 = \omega_0^2 + 4\omega_s^2 \sin^2 \frac{Ka}{2}}$$

Using K^2 assumption,

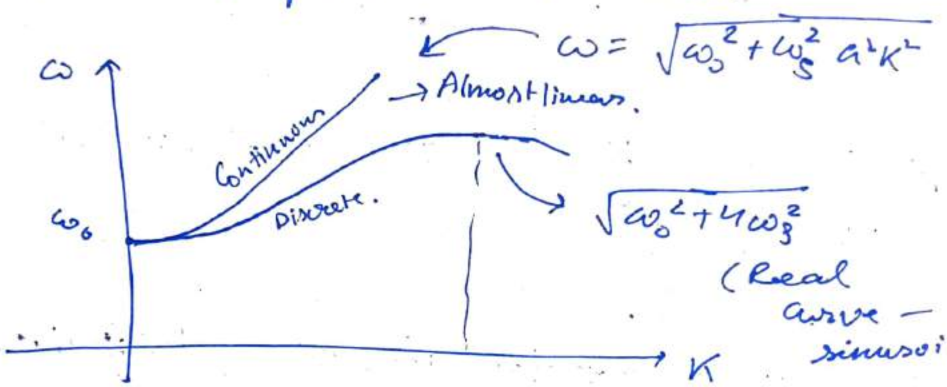
$$\boxed{\omega^2 = \omega_0^2 + \omega_s^2 a^2 K^2} \quad \text{(Small value approx)}$$

As you see, when K^2 is high, it can dominate ω_0^2 term. And when K^2 is small, $\omega^2 \approx \omega_0^2$.

(Usual dispersion relation case)

Plot →

(Dispersion relation)



(*) K → Like momentum

(*) → ω → Like energy

Discussion about how conv. K for e^- in superconductor is constant curve - No energy change even if K changes.

We now look at case,

$\omega^2 < \omega_0^2$ (for K^2 expression)

$$\Rightarrow K^2 = \frac{\omega^2 - \omega_0^2}{\omega_s^2 a^2} < 0$$

$\Rightarrow K \rightarrow$ Imaginary.

$\Rightarrow K = i\kappa$ (Kappa)

$$\kappa = \sqrt{\frac{\omega_0^2 - \omega^2}{\omega_s^2 a^2}}$$

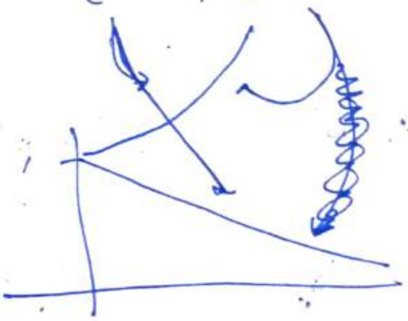
So now,

$f(x) = C\cos Kx + D\sin Kx \longrightarrow D = 0$ (why?)

$= C' e^{-iKx} + D' e^{+Kx}$ (K real here)

↓ otherwise

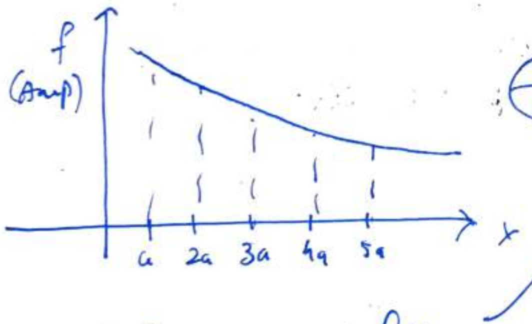
Plotting,



far away oscillator oscillates at larger and larger amp → Who will provide this energy? Non-physical.

So, $\boxed{f(x) = C e^{-Ka}}$.

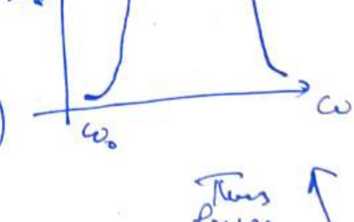
→ Since exp growing term always dominates. (unbounded)



(*) Check Ch3 Ex 9 Crawford for this.

☐ Check the filter analogy

(*) Makes for a good filter.



Rewriting,

$$\omega^2 = \omega_0^2 + 4\omega_s^2 \sin^2\left(\frac{Ka}{2}\right)$$

$$\rightarrow \sin^2 \frac{Ka}{2} = \frac{\omega^2 - \omega_0^2}{4\omega_s^2}$$

Thus filter

Even if $\omega^2 > \omega_0^2 \longrightarrow$ we might get value of sin more than 1 $\Longrightarrow$ K is imaginary (Again)

☐ Verify

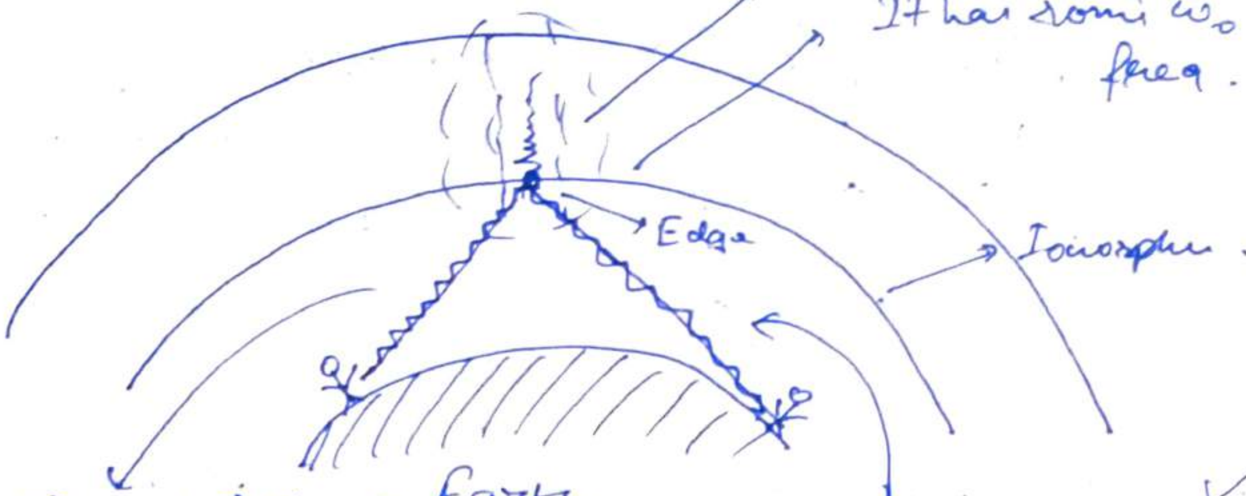
↳ If ω is small enough

So → ① Imaginary if ω is too high, ② Imag if ω is too low

Back to ionosphere.

If $\omega^2 > \omega_0^2 \rightarrow$ Nothing will go out (fizzle out exp decay)

It has some ω_0 (Natural) freq.



$\vec{E} \propto e^{i(\vec{k}\cdot\vec{r} - \omega t)}$
$\downarrow$
forcing freq.

So what happens? Nothing goes out — it must go somewhere.

It reflects (Like TIR)

An optical medium is nothing but a series of e^- as oscillator with damp.

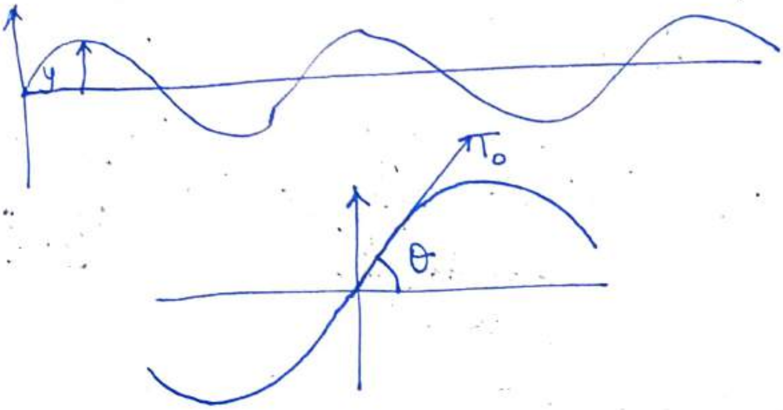
When light falls on it, it oscillates.

29/09/2023

Reflections

$$y(x,t) = C \sin \frac{n\pi x}{L} \cos(\omega t + \phi)$$

$$= \frac{C}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega t + \phi\right) - \sin\left(\frac{n\pi x}{L} - \omega t - \phi\right)\right]$$



at each pt., T_0 is the force suppose,

then $F = T_0 \sin\theta \approx T_0 \tan\theta = \frac{T_0 \partial y}{\partial y}$

$$y = y(x - vt)$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} \quad (z = x - vt)$$

$$= \frac{dy}{dz} \cdot 1 = \frac{dy}{dz}.$$

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{\partial z}{\partial t} = \frac{dy}{dz} \cdot (-v) = -v\frac{dy}{dz}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{v}\frac{dy}{dt}}$$

$$F = T_0 \frac{\partial y}{\partial x}$$

Hence, $\boxed{\frac{dF}{dx} = \frac{-T_0}{v}\frac{dy}{dt}}$

Dimensional Analysis

$$[T_0] = MLT^{-2}$$

$$\left[\frac{T_0}{v}\right] = \frac{MLT^{-2}}{LT^{-1}} = \boxed{MT^{-1}}$$ ⇒ dimensionally similar to velocity dependent damping coeff. (γ)

Impedance (Z)

kind of damping

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

$$\ddot{x} + \underbrace{\frac{\gamma}{m}}_{\alpha}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0$$

$$\text{Impedance}(Z) = \frac{T_0}{v} = \frac{T_0}{\sqrt{T_0/\rho}} = \sqrt{T_0\rho}$$

v of travelling wave

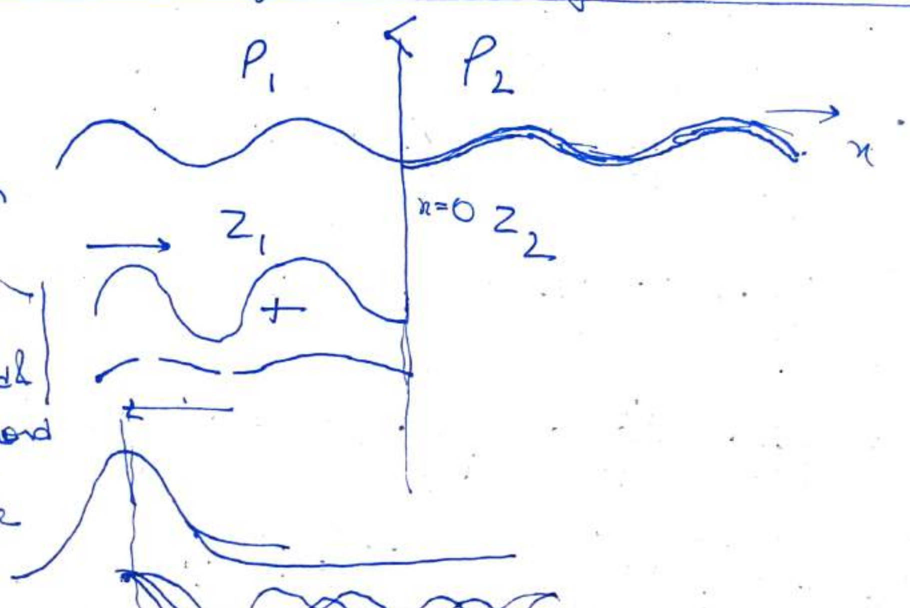
$\boxed{ML^{-1}}$

$$F = -\sqrt{T_0\rho}\,\frac{dy}{dt} = \boxed{-Z\frac{\partial y}{\partial t}}$$

for forward moving wave.

for backward, $v \to \ominus$ve, $F = Z\frac{\partial y}{\partial t}$

Combination of two strings with different ρ



Sum of forward & backward moving wave

↳ except for the beginning, when

all the cos func.s are in same phase, in the other places, they cancel out.

Also, the peak moves with group velocity (v_g)

Forward component (y_{inc})

$$= C\sin(kx - \omega t + \phi) = C\sin k\left(x - vt + \frac{\phi}{k}\right)$$

$$y_{ref} = C\sin k\left(x + vt + \frac{\phi}{k}\right) \quad \text{— (1)}$$

Conditions

(1) $y_{inc} + y_{ref} = y_{tra}$ (y transmitted) ← otherwise string breaks (?)

others

(2) $F = -Z_1 \dfrac{\partial y_{inc}}{\partial t} - Z_1 \dfrac{\partial y_{ref}}{\partial t}$

$F = -Z_2 \dfrac{\partial y_{tra}}{\partial t}$

describe force at the same point → must be equal.

$$\boxed{-Z_1 \frac{\partial y_i}{\partial t} + Z_1 \frac{\partial y_r}{\partial t} = -Z_2 \frac{\partial y_t}{\partial t}} \quad \text{— (2),}$$

$$F_i = -\frac{T_0}{v}\frac{\partial y_i}{\partial t} = -Z_1 \frac{\partial y_i}{\partial t}$$

$$F_r = \frac{T_0}{v}\frac{\partial y_r}{\partial t} = Z_1 \frac{\partial y_r}{\partial t}$$

⇒ from (1) in (2),

$$-Z_1 \frac{\partial y_i}{\partial t} + Z_1 \frac{\partial y_r}{\partial t} = -Z_2 \frac{\partial y_i}{\partial t} - Z_2 \frac{\partial y_r}{\partial t}$$

$$\Rightarrow (Z_1 - Z_2)\frac{\partial y_i}{\partial t} = (Z_1 + Z_2)\frac{\partial y_r}{\partial t}$$

$$\Rightarrow \frac{\partial y_r}{\partial t} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right) \frac{\partial y_i}{\partial t}$$

Integrating, we derive a relation between incident & reflected wave.

$$y_r = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right) y_i + \text{const.}$$

If $y_i = 0$, y_r cannot exist (common sense)

$$\Longrightarrow \boxed{\text{const.} = 0}$$

$$\boxed{y_r = R_{12}\, y_i} \quad \left[R_{12} = \frac{Z_1 - Z_2}{Z_1 + Z_2}\right]$$

Hence, if $\boxed{Z_1 = Z_2,\ \text{Reflection vanishes}}$

if $Z_2 > Z_1$, $R_{12} \to \ominus$ve, $y_{inc} = Ae^{i(kx-\omega t)}$

$$y_{ref} = \frac{-1}{3} y_{inc} = \frac{-1}{3} Ae^{i(kx-\omega t)}$$

$$= \frac{1}{3} Ae^{i(kx-\omega t)} \cdot e^{i\pi} \quad (e^{i\pi} = -1)$$

$$= \frac{1}{3} Ae^{-i\omega t} \cdot e^{i\pi} \quad (\text{at boundary, } x=0)$$

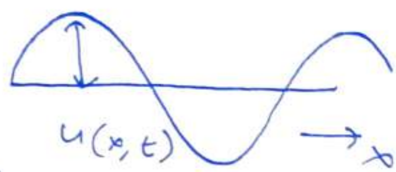
Considering a boundary, where the string is continuous, hence $F_{x-} = F_{x+}$, gives us

$$y_r = \frac{Z_1 - Z_2}{Z_1 + Z_2} y_i$$

©Last page to attach note for last class from the other copy.

11th Oct 2023

$$\boxed{\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$



$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow \text{3 eqns.}$$

3D wave.

$\Rightarrow E_x(x, y, z, t)$, etc.

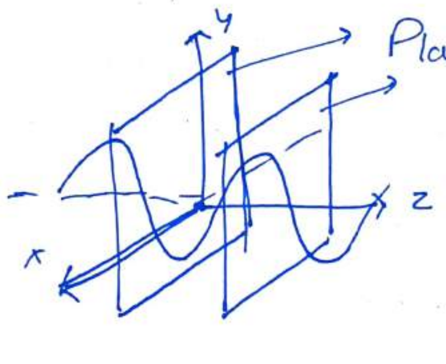
Like ripples in a pond, etc.

Actual light fronts are spherical in nature.

To reduce complexity, we study,

□ Monochromatic plane waves →

$$\vec{E} = e^{i(kz - \omega t)} \rightarrow \text{propagates along z direction}$$



Planes ⊥ to direction of propagation. (same phase on same plane)

and imagine,

$$\vec{B} = B_0 e^{i(kz - \omega t)}$$

Note: Same phase as $\vec{E}$

Ensured by Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} \rightarrow \text{Only } \partial_z \text{ terms survive, as all comps of E have kz dependence.}$$

$\Rightarrow B$

...charge, we have,

$\nabla\cdot\vec{E} = 0$

$(\partial\hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_z)\cdot(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}) = 0$

$\Rightarrow 0 + 0 + ikE_{0_z}e^{i(kz-\omega t)} = 0$

$\Rightarrow \boxed{E_{0z} = 0} \rightarrow$ $\vec{E}$ is purely transverse to direction of propagation.

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & 0 \end{vmatrix}$$

$$= \hat{i}(-\partial_z E_y) - \hat{j}(-\partial_z E_x) + \hat{k}(0)$$

$$= -\hat{i}\,\partial_z E_y + \hat{j}\,\partial_z E_x$$

usual diff

$$= -\hat{i}(ikE_y) + \hat{j}(ikE_x)$$

Now, this should be equal to $-\frac{\partial\vec{B}}{\partial t}$

Also, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{B_{0_z} = 0} \rightarrow$ Same deal.

So, $-\frac{\partial\vec{B}}{\partial t} = -\vec{B_0}(-i\omega)e^{i(kz-\omega t)}$

$$= \vec{B}(i\omega)$$

$$= i\omega(B_x\hat{i} + B_y\hat{j})$$

Thus, we must equate, to get two relations;

$-ikE_y = i\omega B_x \Rightarrow \boxed{B_x = -\frac{k}{\omega}\cdot E_y}$

$ikE_x = i\omega B_y \Rightarrow \boxed{B_y = \frac{k}{\omega}E_x}$

By linear disp relation, $\frac{\omega}{k} = v = c$ (in this case)

$\Rightarrow B_x = -\frac{1}{c}E_y \quad , \quad B_y = \frac{1}{c}E_x$

$$\hat{K} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ E_x & E_y & 0 \end{vmatrix} = -E_y \hat{i} + E_x \hat{j}$$

(Plane aligned)

$$\therefore \boxed{\vec{B} = \frac{1}{c}(\hat{K} \times \vec{E})}$$

↳ $\vec{E}$ is ⊥ to dir of prop and $\vec{B}$

⊛ We know,

$$\boxed{u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)}$$ → Energy density of EM wave

So, to calculate this, we use,

$$B^2 = \frac{1}{c^2}(\underbrace{\hat{K} \times \vec{E}}_{\vec{A}}) \cdot (\underbrace{\hat{K}}_{\vec{B}} \times \underbrace{\vec{E}}_{\vec{C}})$$

→ use $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\rightarrow \vec{B}^2 = \frac{1}{c^2} \underset{A}{\vec{E}} \cdot \left[(\hat{K} \times \vec{E}) \times \hat{K}\right]$$

$$\rightarrow B^2 = \frac{1}{c^2} \vec{E} \left[\vec{E}(\hat{K} \cdot \hat{K}) - \hat{K}(\vec{E} \cdot \hat{K})\right]$$

→ cyclic reorder of scalar triple.

$$\rightarrow \boxed{B^2 = \frac{1}{c^2} E^2}$$

Putting in prev energy density eqn,

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu c^2} E^2\right) = \epsilon E^2$$

$$\rightarrow \boxed{u = \epsilon E^2}$$

Best if we work in scalars here → u is scalar.

$\vec{E} = E_{0x}\, e^{i(kz-\omega t)}\, \hat{i} + E_{0y}\, e^{i(kz-\omega t)}\, \hat{j}$

Obvious from this, if we convert to real trig form, we get, ~~a $\cos^2(kz-\omega t)$~~

$$E^2 = (E_{0x}^2 + E_{0y}^2)\cos^2(\cancel{\omega t}\; kz-\omega t)$$

$$\therefore \langle u \rangle = \frac{1}{T}\int_0^T u\, dt$$

$$= \frac{\epsilon}{T}(E_{0x}^2 + E_{0y}^2)\underbrace{\int_0^T \cos^2(kz-\omega t)\, dt}_{1/2}$$

$$= \frac{1}{2}\epsilon(E_{0x}^2 + E_{0y}^2)$$

$$\Rightarrow \boxed{\langle u \rangle = \frac{1}{2}\epsilon E_0^2}$$

☐ Learn a rough derivation of

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

(*) Poynting vector →

$$\boxed{\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}}$$ → Along direction of propagation.

Calculation,

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\frac{1}{c}(\hat{k}\times\vec{E})$$

$$= \frac{1}{\mu c}\left\{\hat{k}(\vec{E}\cdot\vec{E}) - \vec{E}\,\underbrace{(\vec{E}\cdot\hat{k})}_{0}\right\}$$

$$= c\,\epsilon E^2\,\hat{k}$$

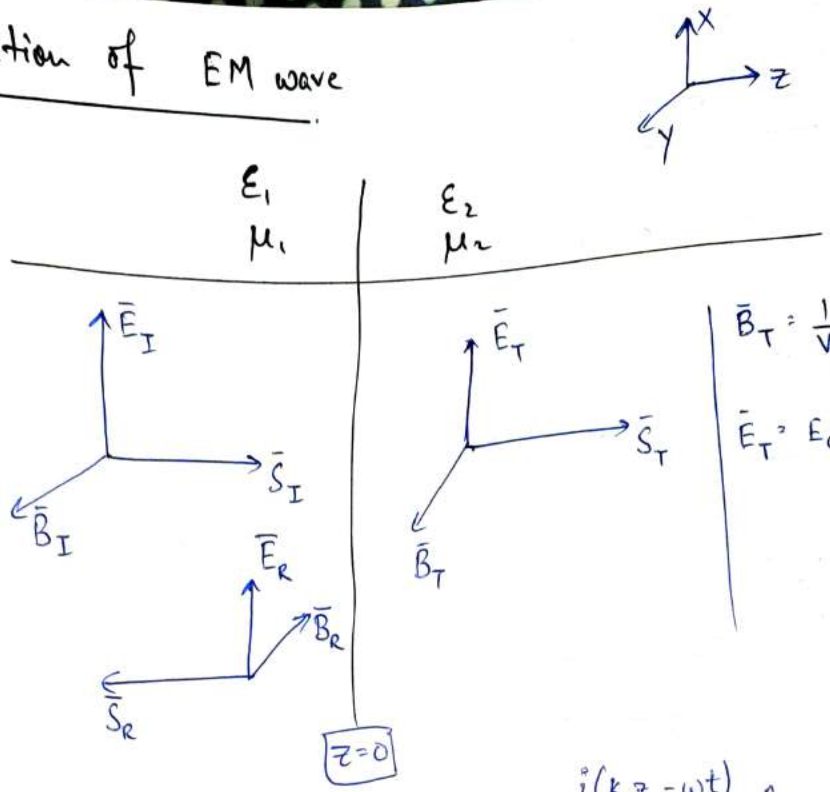
$$\Rightarrow \boxed{\vec{S} = c\,\epsilon E^2\,\hat{k}}$$

Oscillating $\Rightarrow \langle \vec{S}^* \rangle = c\,\langle u \rangle\,\hat{k}$

Intensity

$$\Rightarrow \boxed{\langle \vec{S} \rangle = \frac{1}{2}\epsilon c E_0^2\,\hat{k} = I\,\hat{k}}$$ (*)

Reflection of EM wave



$$\bar{B}_T = \frac{1}{V_2} E_{0T}\, e^{i(k_2 z - \omega t)}\, \hat{j}$$

$$\bar{E}_T = E_{0T}\, e^{i(k_2 z - \omega t)}\, \hat{i}$$

$$\bar{E}_I = E_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{i}$$

$$\bar{E}_R = E_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{i}$$

$$\frac{1}{V_1} E_{0I} \quad \bar{B}_I = B_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{j}$$

$$\frac{1}{V_1} E_{0R} \quad \bar{B}_R = B_{0R}\, e^{i(-k_1 z - \omega t)}\, (-\hat{j})$$

Apply Boundary Conditions.

B3 $\quad E_{0I}\, e^{-i\omega t} + E_{0R}\, e^{-i\omega t} = E_{0T}\, e^{-i\omega t}$

$$\Rightarrow \boxed{E_{0I} + E_{0R} = E_{0T}}$$

B4 $\quad \boxed{\frac{1}{\mu_1}\frac{1}{V_1} E_{0I} - \frac{1}{\mu_1}\frac{1}{V_1} E_{0R} = \frac{1}{\mu_2}\frac{1}{V_2} E_{0T}}$

$$E_{0R} = \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1}\right) E_{0I}$$

$$\Rightarrow \quad E_{0R} = \left(\frac{1 - \frac{\mu_1 v_1}{\mu_2 v_2}}{1 + \frac{\mu_1 v_1}{\mu_2 v_2}}\right) E_{0I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0I}$$

$$= \left(\frac{1-\beta}{1+\beta}\right) E_{0I} \quad , \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\boxed{E_{0T} = \left(\frac{2}{1+\beta}\right) E_{0I}}$$

$$I = \frac{1}{2} \varepsilon v E_0^2 \ \hat{k}$$

$$R = \frac{I_R}{I_I} = \frac{\frac{1}{2} \varepsilon_1 v_1 E_{0R}^2}{\frac{1}{2} \varepsilon_1 v_1 E_{0I}^2} = \left(\frac{1-\beta}{1+\beta}\right)^2$$

Reflection coefficient.

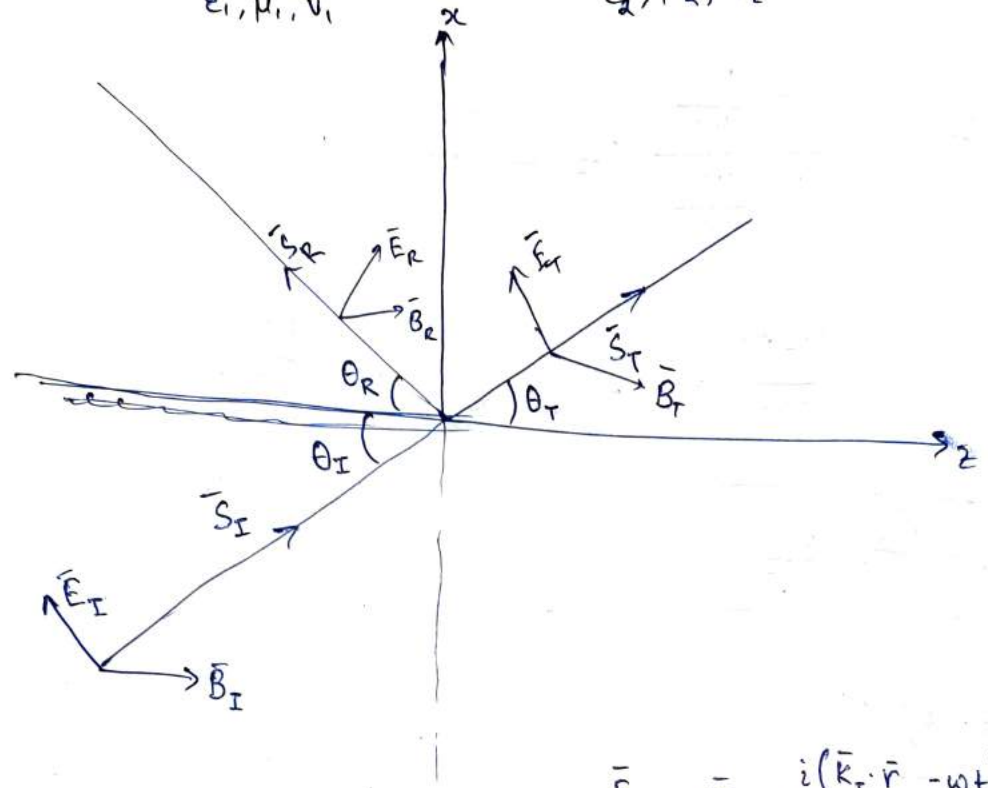
$$T = \frac{I_T}{I_I} = \frac{\frac{1}{2} \varepsilon_2 v_2 E_{0T}^2}{\frac{1}{2} \varepsilon_1 v_1 E_{0I}^2}$$

$$\frac{1}{v^2} = \mu \varepsilon$$

$$= \frac{\mu_1 v_1^2 \cancel{v_2}}{\mu_2 v_2^{\cancel{2}} \cancel{v_1}} \left(\frac{4}{(1+\beta)^2}\right)$$

$$= \frac{4\beta}{(1+\beta)^2}$$

Refraction of EM-Waves



$$\bar{E}_I = \bar{E}_{0I}\, e^{i(\bar{k}_I\cdot\bar{r} - \omega t)}$$

$$\bar{E}_R = \bar{E}_{0R}\, e^{i(\bar{k}_R\cdot\bar{r} - \omega t)}$$

$$\bar{E}_T = \bar{E}_{0T}\, e^{i(\bar{k}_T\cdot\bar{r} - \omega t)}$$

B3 $\rightarrow$ $\boxed{\text{at } z=0}$.

$$\bar{E}_{0I}\, e^{i(\bar{k}_I\cdot\bar{r} - \omega t)} + \bar{E}_{0R}\, e^{i(\bar{k}_R\cdot\bar{r} - \omega t)} = \bar{E}_{0T}\, e^{i(\bar{k}_e\cdot\bar{r} - \omega t)}$$

$$\bar{E}_{0I}\, e^{i(\bar{k}_I\cdot\bar{r})} + \bar{E}_{0R}\, e^{i(\bar{k}_R\cdot\bar{r})} = \bar{E}_{0T}\, e^{i(\bar{k}_T\cdot\bar{r})}$$

$$\bar{K}_I\cdot\bar{r} = \bar{K}_R\cdot\bar{r} = \bar{K}_T\cdot\bar{r}$$

$$x K_{Ix} + y K_{Iy} = x K_{Rx} + y K_{Ry} = x K_{Tx} + y K_{Ty}$$

valid for all x, y.

$$\Rightarrow \boxed{K_{Ix} = K_{Rx} = K_{Tx}} \;\&\; \boxed{K_{Iy} = K_{Rx} = K_{Tx}}$$

By choosing appropriate coordinate system we can make $k_{Iy} = 0$. $\Rightarrow k_{Ry} = k_{Ty} = 0$.

$\Rightarrow$ All the three waves are in the same plane.

$$\Rightarrow \left.\begin{aligned} k_{Ix} &= k_I \sin\theta_I \\ k_{Rx} &= k_R \sin\theta_R \end{aligned}\right\} \qquad \boxed{k_I = k_R} \quad \frac{2\pi}{d} = \frac{2\pi}{d}$$

$$\Rightarrow \boxed{\theta_I = \theta_R}$$

$$\Rightarrow \left.\begin{aligned} k_{Ix} &= k_I \sin\theta_I \\ k_{Tx} &= k_T \sin\theta_T \end{aligned}\right\} \quad \left| \quad \left.\begin{aligned} k_I &= \frac{2\pi}{d_1} = \frac{2\pi\nu}{v_1} \\ k_T &= \frac{2\pi}{d_2} = \frac{2\pi\nu}{v_2} \end{aligned}\right\} \Rightarrow \boxed{v_1 k_I = v_2 k_T}\right.$$

$$k_I \sin\theta_I = k_T \sin\theta_T$$

$$\Rightarrow \boxed{\frac{k_I}{k_T} = \frac{\sin\theta_T}{\sin\theta_I} = \frac{v_2}{v_1} = \frac{n_1}{n_2}}$$

$$\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$$

Electromagnetic waves in matter

For linear and homogeneous medium (no free charge or current) the Maxwell's relations are

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

$\mu \rightarrow$ permeability

$\epsilon \rightarrow$ permittivity

In vacuum

$\epsilon \rightarrow \epsilon_0 = 8.85 \times 10^{-12}\ C^2 N^{-1} m^{-2}$

$\mu \rightarrow \mu_0 = 4\pi \times 10^{-7}\ C^{-2}\ N\ T^2$

$\epsilon = \epsilon_r\epsilon_0$, where $\epsilon_r \rightarrow$ dielectric constant

$\mu \sim \mu_0$, for most linear, homogeneous media

Linear: $\vec{P}$ = Polarization (induced) $= \epsilon_0\chi_e\vec{E}$ ← linear on $\vec{E}$

$\vec{M}$ = Magnetization (induced) $= \mu_0\chi_m\vec{H}$ ← linear on $\vec{H}$

Homogeneous: ϵ and μ do not depend on $\vec{r}$.

$\mu_0(\vec{H}+\vec{M}) = \vec{B}$

$\Rightarrow \mu_0(1+\chi_m)\vec{H} = \vec{B}$

$\Rightarrow \vec{H} = \frac{1}{\mu}\vec{B}$

with $\mu = \mu_0(1+\chi_m)$

To construct the wave equation we use,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

As, $\vec{\nabla}\cdot\vec{E} = 0$, we have,

$$\boxed{\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}} \Rightarrow \boxed{c^2 = \frac{1}{\mu\epsilon}}$$

Also,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu\epsilon\frac{\partial\vec{E}}{\partial t}\right) = \mu\epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2}} \Rightarrow \boxed{c^2 = \frac{1}{\mu\epsilon}}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = ?$$

$$\vec{\nabla} \times \vec{A} \equiv \epsilon_{ijk}\, \partial_j A_k \quad \text{using Levi-Civita symbols}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \epsilon_{ijk}\, \partial_j\, \epsilon_{klm}\, \partial_l A_m$$

$$= \epsilon_{ijk}\, \epsilon_{klm}\, \partial_j \partial_l A_m$$

$$= \epsilon_{kij}\, \epsilon_{klm}\, \partial_j \partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \partial_j \partial_l A_m$$

$$= \partial_j \partial_i A_j - \partial_j^2 A_i$$

$$= \partial_i \partial_j A_j - \partial_j^2 A_i$$

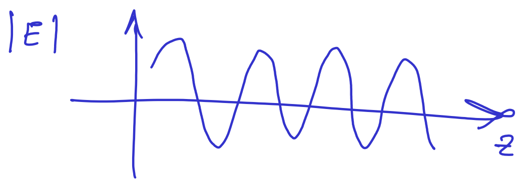
$$\equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Consider monochromatic plane wave solutions.

$$\vec{E} = \vec{E}_0\, e^{i(kz - \omega t)} = \vec{E}_0\, f(z,t)$$

and

$$\vec{B} = \vec{B}_0\, e^{i(kz - \omega t)} = \vec{B}_0\, f(z,t)$$



[Note: no phase lag between $\vec{E}$ and $\vec{B}$, as dictated by Faraday's law]

Now, $\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow E_{0z} = 0$

and $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow B_{0z} = 0$

The waves are transverse!

Also, $\vec{\nabla} \times \vec{E}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_{0x} f(z,t) & E_{0y} f(z,t) & 0 \end{vmatrix}$$

$$= \hat{i}\left(-ik\, E_{0y}\, f(z,t)\right) + \hat{j}\left(+ik\, E_{0x}\, f(z,t)\right) = -\frac{\partial \vec{B}}{\partial t}$$

$$= \hat{i}\left(+i\omega\, B_{0x}\, f(z,t)\right) + \hat{j}\left(+i\omega\, B_{0y}\, f(z,t)\right)$$

$$\Rightarrow \left.\begin{aligned} -k E_{0y} &= \omega B_{0x} \\ \text{and} \quad k E_{0x} &= \omega B_{0y} \end{aligned}\right\} \Rightarrow \vec{B} = \frac{k}{\omega}(\hat{k} \times \vec{E}_0)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ E_{0x} & E_{0y} & 0 \end{vmatrix} = -E_{0y}\,\hat{i} + E_{0x}\,\hat{j}$$

So, we have,

$$\vec{B}_0 = \frac{1}{c}\,\hat{k} \times \vec{E}_0 \Rightarrow B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

$$\Rightarrow \vec{B} = \frac{1}{c}\big(\underset{\substack{\uparrow \\ \text{direction of propagation}}}{\hat{k}} \times \vec{E}\big)$$

We write $\vec{B}$ in terms of $\vec{E}$ and help reduce the complexity of the situation.

Energy density and flux (intensity)

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$$

Use,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\begin{aligned}
B^2 &= \frac{1}{c^2}(\hat{k}\times\vec{E})\cdot(\hat{k}\times\vec{E}) \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left[(\hat{k}\times\vec{E})\times\hat{k}\right] \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left[\hat{k}\times(\vec{E}\times\hat{k})\right] \\
&= \frac{1}{c^2}\,\vec{E}\cdot\left\{(\hat{k}\cdot\hat{k})\vec{E} - \cancelto{=0}{(\hat{k}\cdot\vec{E})}\,\hat{k}\right\} \\
&= \frac{1}{c^2} E^2
\end{aligned}$$

$$\Rightarrow u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}\frac{E^2}{c^2}\right) = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}\cdot\epsilon\mu E^2\right) = \epsilon E^2$$

E executes sinusoidal oscillation in time.

$$\Rightarrow \langle u \rangle = \langle \epsilon E^2 \rangle = \frac{1}{2}\epsilon E_0^2$$

Now, Poynting vector is

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}\vec{E}\times\frac{1}{v}(\hat{k}\times\vec{E}) \\
&= \frac{1}{\mu v}\left\{\hat{k}\,\vec{E}\cdot\vec{E} - \vec{E}(\vec{E}\cdot\hat{k})\right\} \\
&= \epsilon v E^2\,\hat{k}
\end{aligned}$$

$$\mu = \frac{1}{\epsilon v^2}$$

Energy flux is

$$\langle \vec{S} \rangle = v\epsilon\langle E^2\rangle\,\hat{k} = \frac{1}{2}\epsilon v E_0^2\,\hat{k} = \underset{\substack{\uparrow \\ \text{intensity}}}{I}\,\hat{k}$$

Boundary conditions

We begin with the Maxwell's equations in the integral form.

(i) $\oint_S \vec{D} \cdot d\vec{a} = Q_{free,\, enclosed}$

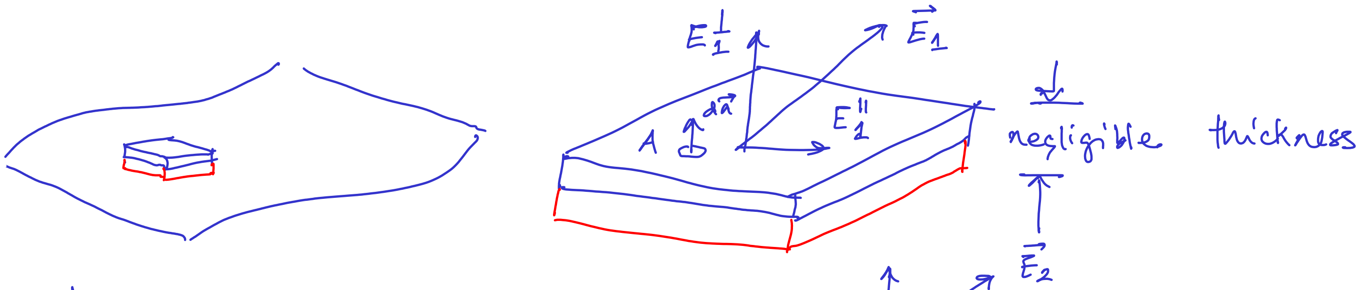
(ii) $\oint_S \vec{B} \cdot d\vec{a} = 0$

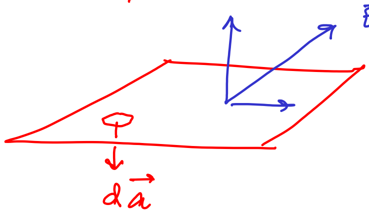
(iii) $\oint_P \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}$

(iv) $\oint_P \vec{H} \cdot d\vec{l} = I_{free,\, enclosed} + \frac{d}{dt}\int_S \vec{D} \cdot d\vec{a}$

From, (i)

We use a thin Gaussian box around the boundary.





$$\epsilon_1 E_1^\perp A - \epsilon_2 E_2^\perp A = 0$$

$$\Rightarrow \boxed{\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp} \quad \text{---} \quad (B1)$$

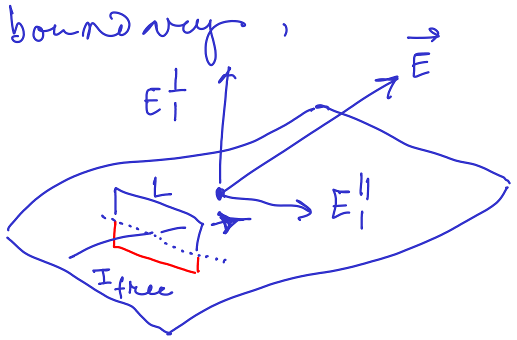
From (ii),

Exactly the same way, from (i) above we get,

$$\boxed{B_1^\perp = B_2^\perp} \quad \text{----} \quad (B2)$$

From (iii)

We use a narrow rectangular loop around the boundary,



$$E_1^\parallel L - E_2^\parallel L = 0$$

For narrow loop the flux vanishes.

$$\Rightarrow \boxed{E_1^\parallel = E_2^\parallel} \quad \text{---} \quad (B3)$$

From (iv),

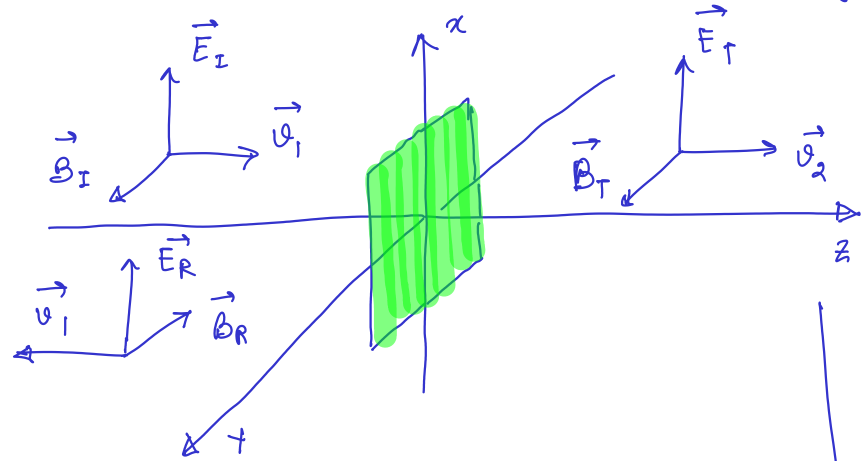
we use the narrow loop above to get,

$$\frac{1}{\mu_1} B_1^{\parallel} L - \frac{1}{\mu_2} B_2^{\parallel} L = 0$$ no free current and using (i) (no free charge).

$$\Rightarrow \boxed{\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_1^{\parallel}} \quad \text{---- (B4)}$$

We shall use (B1) to (B4) to get the laws of geometric optics.

Consider the following boundary



$$\vec{E}_I = E_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{i}$$

$$\vec{B}_I = B_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{j} = \frac{1}{v_1} E_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{j}$$

$$\vec{E}_R = E_{0R}\, e^{i(k_1 z - \omega t)}\, \hat{i}$$

$$\vec{B}_R = -\frac{1}{v_1} E_{0I}\, e^{i(-k_1 z - \omega t)}\, \hat{j}$$

and similarly $\vec{E}_T$ & $\vec{B}_T$

(B1) and (B2) is trivially zero.

From (B3) we get,

$$E_{0I} + E_{0R} = E_{0T}$$

and from (B4) we get,

$$\frac{1}{\mu_1}\left(\frac{1}{v_1} E_{0I} - \frac{1}{v_1} E_{0R}\right) = \frac{1}{\mu_2}\frac{1}{v_2} E_{0T} = \frac{1}{\mu_2 v_2}\left(E_{0I} + E_{0R}\right)$$

$$\Rightarrow \left(\frac{1}{\mu_1 v_1} + \frac{1}{\mu_2 v_2}\right) E_{0R} = \left(\frac{1}{\mu_1 v_1} - \frac{1}{\mu_2 v_2}\right) E_{0I}$$

$$\Rightarrow E_{0R} = \frac{1 - \frac{\mu_1 v_1}{\mu_2 v_2}}{1 + \frac{\mu_1 v_1}{\mu_2 v_2}} \cdot E_{0I} = \frac{1-\beta}{1+\beta} E_{0I}$$

where, $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$ | using, $\dfrac{n_1}{n_2} = \dfrac{v_2}{v_1}$

$$\Rightarrow \quad E_{0T} = (1+R)E_{0I} = \left(1 + \frac{1-\beta}{1+\beta}\right)E_{0I} = \left(\frac{2}{1+\beta}\right)E_{0I}$$

Intensity of a light beam

$$= \boxed{\frac{1}{2}\epsilon v E_0^2}$$

$$\Rightarrow \quad R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{1-\beta}{1+\beta}\right)^2 \sim \left(\frac{1-n_2/n_1}{1+n_2/n_1}\right)^2$$

as $\mu \approx \mu_0$ for most materials

$$= \left(\frac{n_1-n_2}{n_1+n_2}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2 E_{0T}^2}{\epsilon_1 v_1 E_{0I}^2} = \frac{n_2^2 n_1}{n_1^2 n_2}\frac{(2n_1)^2}{(n_1+n_2)^2}$$

$$= \frac{4n_1 n_2}{(n_1+n_2)^2}$$

$$\frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_1\mu}{\epsilon_2\mu} = \frac{v_2^2}{v_1^2} = \frac{n_1^2}{n_2^2}$$

So, $R + T = 1$.

For, $n_1 = 1$, $n_2 = 1.5$, $R = 0.04$ and $T = 0.96$.

<u>Wavevector</u>

We have $\vec{E} = \vec{E}_0 \, e^{i(kz - \omega t)}$

phase

$kz \equiv k\hat{k} \cdot z\hat{k} \longrightarrow$ assumption the EM wave is propagating along z.

In general, we have

instead of $k\hat{k} \to k_x\hat{i} + k_y\hat{j} + k_z\hat{k} = \vec{k} = k\hat{k}$

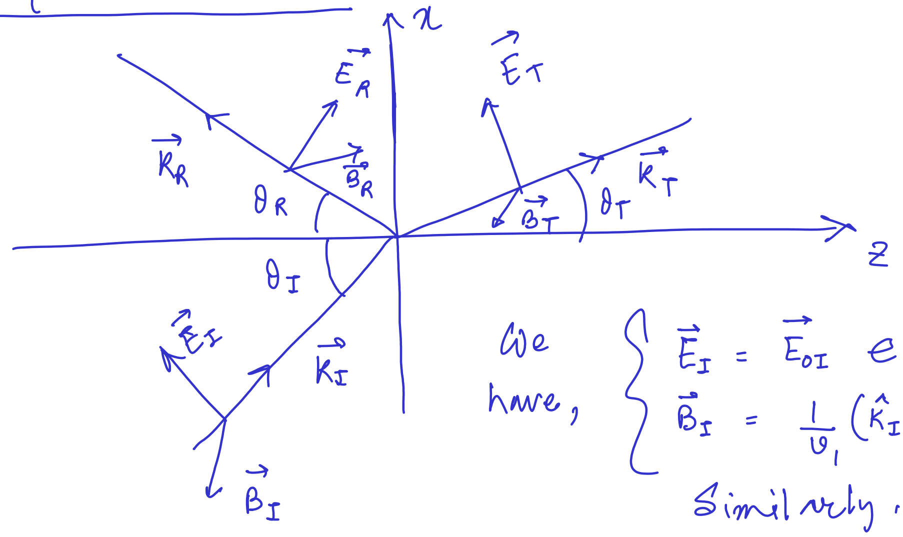
and $z\hat{k} \to x\hat{i} + y\hat{j} + z\hat{k} = \vec{r} = r\hat{r}$

$$\Rightarrow \quad \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \vec{E}_0 \, e^{i(k\hat{k}\cdot\vec{r} - \omega t)}$$

and $\boxed{v = \frac{\omega}{k}}$

Oblique incidence



We have,

$$\begin{cases} \vec{E}_I = \vec{E}_{0I}\, e^{i(\vec{K}_I \cdot \vec{r} - \omega t)} \\ \vec{B}_I = \dfrac{1}{v_1}(\hat{K}_I \times \vec{E}_{0I})\, e^{i(\vec{K}_I \cdot \vec{r} - \omega t)} \end{cases}$$

Similarly, for R and T

We note,

$$k_I = \frac{\omega}{v_1} = k_R \quad \text{and} \quad K_T = \frac{\omega}{v_2} = \frac{v_1}{v_2} k_I = \frac{n_2}{n_1} k_I$$

Matching boundary conditions will lead to

$$(\ )_I\, e^{i(\vec{K}_I \cdot \vec{r} - \omega t)} + (\ )_R\, e^{i(\vec{K}_R \cdot \vec{r} - \omega t)} = (\ )_T\, e^{i(\vec{K}_T \cdot \vec{r} - \omega t)} \quad \text{at } z = 0$$

The oscillatory part must match on both sides.

$$\Rightarrow \vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \quad \text{when } z = 0$$

$$\Rightarrow x K_{Ix} + y K_{Iy} = x K_{Rx} + y K_{Ry} = x K_{Tx} + y K_{Ty}$$

$$\Rightarrow \text{For } y = 0, \quad K_{Ix} = K_{Rx} = K_{Tx}$$

$$\text{and for } x = 0, \quad K_{Iy} = K_{Ry} = K_{Ty}$$

Now, if we set $K_{Iy} = 0$ (choosing an axes for a given incidence),

we have $K_{Ry} = K_{Ty} = 0$

$\Rightarrow$ $\vec{K}$ vectors are all on a single (xz in this case) plane $\longrightarrow$ plane of incidence.

<u>First law</u>

Now, $k_{Ix} = K_{Rx} = K_{Tx}$

$$\Rightarrow k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

$\Rightarrow$ $\theta_I = \theta_R$ as $k_I = k_R$ $\longrightarrow$ Second law / law of reflection

and $\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{K_I}{K_T} = \dfrac{n_1}{n_2}$ $\longrightarrow$ Third law / Snell's law.

Boundary conditions

(B1) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \rightarrow$ for z

(B2) $B_1^{\perp} = B_2^{\perp} \rightarrow$ for z

(B3) $E_1^{\parallel} = E_2^{\parallel} \rightarrow$ for both x, y

(B4) $\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \rightarrow$ for both x, y

for the chosen boundary

(B1) $\Rightarrow \epsilon_1 \left(- E_{0I} \sin\theta_I + E_{0R} \sin\theta_R\right) = \epsilon_2 \left(- E_{0T} \sin\theta_T\right)$ .... (1)

(B2) $\Rightarrow 0 = 0$

(B3) $\Rightarrow \left(E_{0I} \cos\theta_I + E_{0R} \cos\theta_R\right) = E_{0T} \cos\theta_T$ .... (2)

(B4) $\Rightarrow \frac{1}{\mu_1}\frac{1}{v_1}\left(E_{0I} - E_{0R}\right) = \frac{1}{\mu_2}\frac{1}{v_2} E_{0T}$ .... (3)

(1) $\Rightarrow E_{0I} - E_{0R} = \frac{\epsilon_2}{\epsilon_1}\frac{\sin\theta_T}{\sin\theta_I} E_{0T} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} E_{0T} = \frac{\mu_1 v_1}{\mu_2 v_2} E_{0T}$ $\quad \epsilon_1 = \frac{1}{\mu_1 v_1^2}$

$= \beta E_{0T}$

(3) $\Rightarrow E_{0I} - E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2} E_{0T} = \beta E_{0T}$

(2) $\Rightarrow E_{0I} + E_{0R} = \frac{\cos\theta_T}{\cos\theta_I} E_{0T} = \alpha E_{0T}$

$$\boxed{E_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0I}, \quad E_{0T} = \left(\frac{2}{\alpha + \beta}\right) E_{0I}}$$

Fresnel's equations

* Transmitted beam $\rightarrow$ always in-phase with the incident beam
* Reflected beam $\rightarrow$ either in-phase or out-of-phase with the incident beam

$\alpha = \beta \Rightarrow$ No reflection condition!

$$\boxed{\sin^2\theta_B = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}}$$

$$\boxed{\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}}{\cos\theta_I}}$$

$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$

$n_1 \sin\theta_I = n_2 \sin\theta_T$

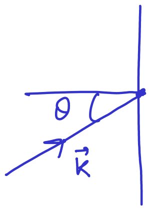
$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$

$\Rightarrow \cos\theta_T = \sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_I\right)^2}$

Intensity

$I \propto \frac{1}{2} \epsilon v E^2 \cos\theta$

$\cos\theta$ ↑ Component of the beam perpendicular to the boundary



Intensity of the reflected wave,

$$\frac{I_R}{I_I} = \frac{\epsilon_1 v_1 E_{0R}^2}{\epsilon_1 v_1 E_{0I}^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

Intensity of the transmitted wave,

$$\frac{I_T}{I_I} = \frac{\epsilon_1 v_1 E_{0T}^2 \cos\theta_T}{\epsilon_2 v_2 E_{0I}^2 \cos\theta_I} = \frac{\mu_1 v_1}{\mu_2 v_2} \cdot \frac{\cos\theta_T}{\cos\theta_I} \cdot \left(\frac{2}{1+\beta}\right)^2 = \beta \cdot \alpha \cdot \frac{4}{(1+\beta)^2}$$

* **Brewster's angle**

$\alpha = \beta \Rightarrow$ no reflection

$$\Rightarrow \frac{\cos\theta_T^0}{\cos\theta_I^0} = \frac{\mu_1 v_1}{\mu_2 v_2} \approx \frac{n_2}{n_1}$$

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} = \frac{1}{\beta}$$

$$\Rightarrow \frac{\sqrt{1-\sin^2\theta_T^0}}{\cos\theta_I^0} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I^0}}{\cos\theta_I^0} = \frac{n_2}{n_1} = \beta$$

$$\Rightarrow \sin^2\theta_I^0 = \frac{\beta^2}{1+\beta^2} \Rightarrow \tan\theta_I^0 = \beta = \frac{n_2}{n_1} \Rightarrow \theta_I^0 = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

Total internal reflection

For $n_2 > n_1$, $\theta_T < \theta_I$

For $n_2 < n_1$, $\theta_T > \theta_I$

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

For a particular θ_I^0 for $n_2 < n_1$,

if $\sin\theta_T^0 = 1$ i.e. $\theta_T^0 = \pi/2$, we have,

$$\sin\theta_I^0 = \frac{n_2}{n_1} \cdot \sin\theta_T^0 = \frac{n_2}{n_1} \Rightarrow \theta_I^0 = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

We get this again from

$$\frac{I_T}{I_I} = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$ with $$\alpha = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)\sin^2\theta_I}}{\cos\theta_I} = 0$$

For α to be real,

$$\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I \leq 1$$

or $\theta_I \leq \sin^{-1}\left(\frac{n_2}{n_1}\right)$

Equality $\longrightarrow$ threshold for <u>total internal reflection</u>

For all $\theta_I > \theta_I^0$, we have "exponential wave" instead of a regular transmission.

Also,

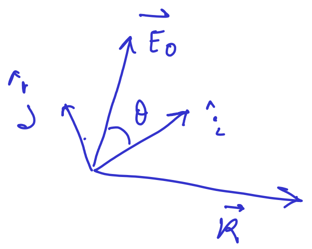
$$\frac{E_{0R}}{E_{0I}} = \frac{\alpha-\beta}{\alpha+\beta} = \frac{\frac{\cos\theta_T}{\cos\theta_I} - \frac{n_2}{n_1}}{\frac{\cos\theta_T}{\cos\theta_I} + \frac{n_2}{n_1}} = \left(\frac{n_1\cos\theta_T - n_2\cos\theta_I}{n_1\cos\theta_T + n_2\cos\theta_I}\right)$$

<u>For "p-type" polarization</u>

Polarization

Consider the plane formed by the displacement vector and the wavevector $\vec{k}$. If the displacement vector remains in the same direction as one moves along $\vec{k}$, we have a linearly polarized light.

$\vec{E} = \vec{E}_0 \cos(kz - \omega t)$ $\qquad$ $\vec{E}_0$ is not a function of z or t.



We choose the axes as shown in the figure.

$$\Rightarrow \vec{E}_0 = E_0 \cos\theta\, \hat{i} + E_0 \sin\theta\, \hat{j}$$

$$\text{and } \vec{E} = \underbrace{\left(E_0 \cos\theta\, \hat{i} + E_0 \sin\theta\, \hat{j}\right)}_{\text{amplitude.}} \cos(kz - \omega t)$$

Where we choose x axis such that xz is the plane of incidence.

$$= \left(E_0^p\, \hat{i} + E_0^s\, \hat{j}\right) \cos(kz - \omega t)$$

amplitude of $\left\{ \begin{array}{l} p\text{-type} \\ p\text{-polarized} \\ p\text{- component} \end{array} \right.$ $\left\{ \begin{array}{l} s\text{-type} \\ s\text{-polarized} \\ s\text{- component} \end{array} \right.$

Continued in the next page.

Now, Fresnel's equations

$$\frac{E_R^p}{E_I^p} = \frac{\alpha - \beta}{\alpha + \beta} \qquad \frac{E_T^p}{E_I^p} = \frac{2}{\alpha + \beta}$$

$$\boxed{\frac{E_R^s}{E_I^s} = \frac{1 - \alpha\beta}{1 + \alpha\beta} \qquad \frac{E_T^s}{E_I^s} = \frac{2}{\alpha + \beta}}$$

not done in the class but is quite straight forward.

Fresnel's equations along with the decomposition to "s" and "p" types provide complete description.

* <u>Case</u> $\alpha = \beta \Rightarrow E_R^p = 0$ and $E_R^s \neq 0$ → <u>Brewster's angle</u>

$\Rightarrow$ Reflected light is linearly polarized.

How polarizers work:

<u>Explanation</u> based on free electrons in "wire".

Now, consider

$$\vec{E} = E_0\, \hat{i} \cos(kz - \omega t) = \frac{E_0}{2}\left\{\hat{i}\cos(kz-\omega t) + \hat{j}\sin(kz-\omega t)\right\} + \frac{E_0}{2}\left\{\hat{i}\cos(kz-\omega t) - \hat{j}\sin(kz-\omega t)\right\}$$

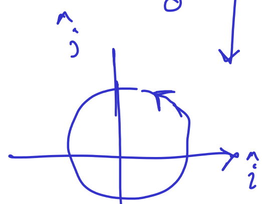
$$= \frac{E_0}{2}\left\{\hat{i}\cos(kz-\omega t) + \hat{j}\cos(kz-\omega t - \pi/2)\right\}$$
$$+ \frac{E_0}{2}\left\{\hat{i}\cos(kz-\omega t) + \hat{j}\cos(kz-\omega t + \pi/2)\right\}$$

using complex notation

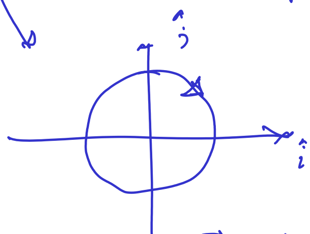
$$= \frac{E_0}{2}\left\{\hat{i}\, e^{i(kz-\omega t)} + \hat{j}\, e^{-i\pi/2} e^{i(kz-\omega t)}\right\}$$
$$+ \frac{E_0}{2}\left\{\hat{i}\, e^{i(kz-\omega t)} + \hat{j}\, e^{+i\pi/2} e^{i(kz-\omega t)}\right\}$$

$$= \frac{E_0}{2}\left\{(\hat{i} + i\hat{j})\, e^{i(kz-\omega t)}\right\}$$
$$+ \frac{E_0}{2}\left\{(\hat{i} - i\hat{j})\, e^{i(kz-\omega t)}\right\}$$

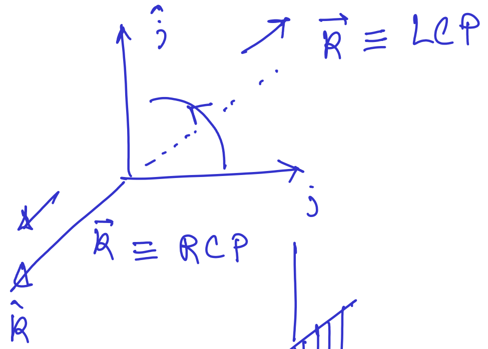
$$= \vec{E}_+ + \vec{E}_-$$

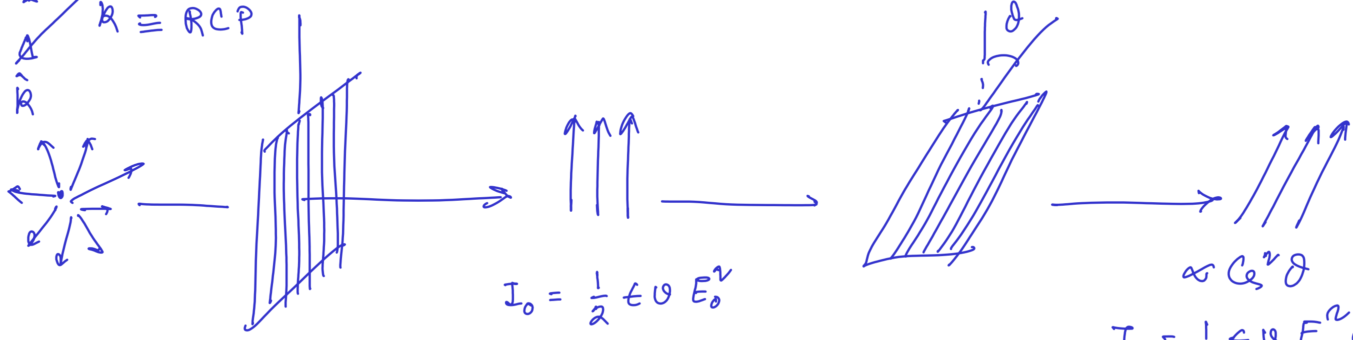


tip of $\vec{E}$ field at a given t as one moves along z → RCP



tip of $\vec{E}$ field at a given t as one moves along z → LCP





(*)

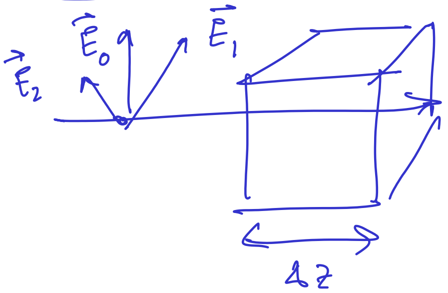
$I_0 = \frac{1}{2} \epsilon v E_0^2$

$I = \frac{1}{2} \epsilon v E_0^2 \cos^2\theta$

Malus's law.

(*) Reflection of RCP gives LCP

(*) Retardation plates (has two optical axes with diff. n values).



$$\frac{\omega}{k} = v = \frac{c}{n} \Rightarrow k = n\frac{\omega}{c}$$

$$\Rightarrow \text{Phase lag} = k\Delta z = n\frac{\omega}{c}\Delta z$$

Lag between two axes

$$= n_1 \frac{\omega}{c}\Delta z - n_2 \frac{\omega}{c}\Delta z$$

$$= \Delta n \frac{\omega}{c} \cdot \Delta z = \Delta n \frac{2\pi}{\lambda_0} \cdot \Delta z$$

$$\boxed{\phi = 2\pi \cdot \Delta n \cdot \frac{\Delta z}{\lambda_0}}$$

$\phi \rightarrow \pi/2 \rightarrow$ quarterwave plate $\rightarrow$ linear to circular

$\phi \rightarrow \pi \rightarrow$ half-wave plate $\rightarrow$ RCP $\rightleftarrows$ LCP

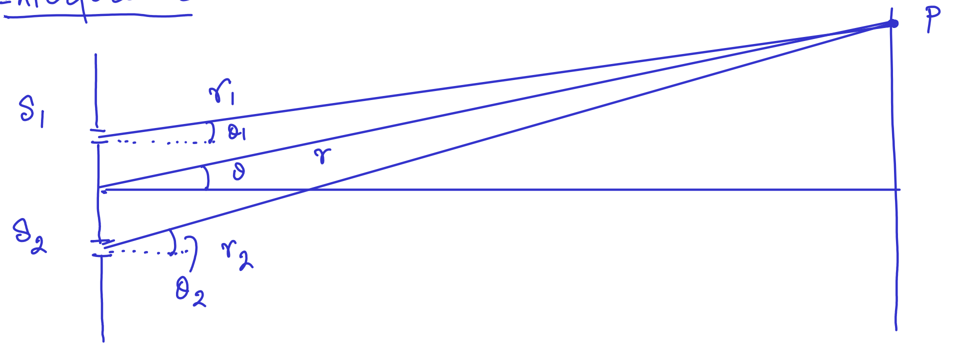
Interference and diffractions

Interference → Multiple discrete sources

diffractions → Continuous source

Both rely on the superposition principle.

Interference



S_1, S_2 are sources. Both emit EM waves with the same amplitude A.

EM waves from S_1 and S_2 arrive at P.

E field at P is given by,

$$E = A e^{i(kr_1 - \omega t + \phi_1)} + A e^{i(kr_2 - \omega t + \phi_2)}$$

$$= A e^{-i\omega t}\left[e^{i(kr_1 + \phi_1)} + e^{i(kr_2 + \phi_2)}\right]$$

$$= A e^{-i\omega t} e^{i\left(k\frac{r_1+r_2}{2} + \frac{\phi_1+\phi_2}{2}\right)}\left[e^{i\left(k\cdot\frac{r_1-r_2}{2} + \frac{\phi_1-\phi_2}{2}\right)} + e^{-i\left(k\frac{r_1-r_2}{2} + \frac{\phi_1-\phi_2}{2}\right)}\right]$$

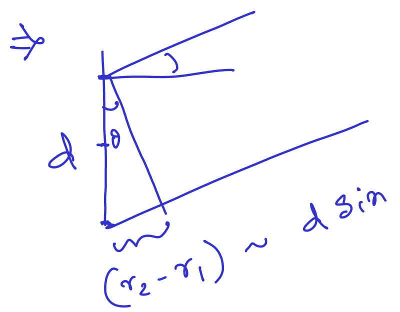
$$= A e^{-i\omega t} e^{i(kr + \phi_{av})}\, 2\cos\left(k\frac{\Delta r}{2} + \frac{1}{2}\Delta\phi\right)$$

$\Delta\phi = \phi_1 - \phi_2$

$\Delta r = r_1 - r_2$

Let, $\theta_1 \sim \theta_2 \sim \theta \implies$ Far-field approx

$\Rightarrow$



$$\text{Intensity} \propto |E|^2$$

$$\propto 4A^2 \cos^2\left(\frac{1}{2}kd\sin\theta + \frac{1}{2}\Delta\phi\right)$$

For coherent sources, $\Delta\phi = 0$

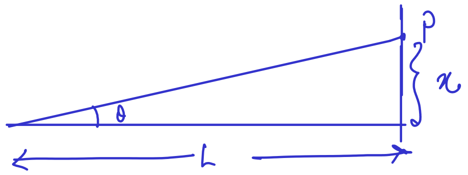
$$\Rightarrow \text{Intensity} \propto 4A^2\cos^2\left(\pi\frac{d}{\lambda}\sin\theta\right)$$

In general, minima occur when,

$$\frac{1}{2}kd\sin\theta = \boxed{\pi\frac{d}{\lambda}\sin\theta = (2n+1)\frac{\pi}{2}} \qquad n = 0, 1, 2, \dots$$

maxima occur when,

$$\frac{1}{2}kd\sin\theta = \boxed{\pi\frac{d}{\lambda}\sin\theta = n\pi} \qquad n = 0, 1, 2, \dots$$



For small θ,

$$\sin\theta \sim \tan\theta \sim \theta \sim \frac{x}{L}$$

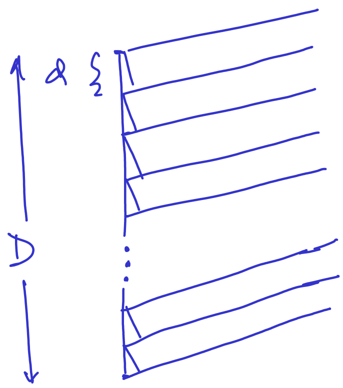
$$\Rightarrow \text{Intensity} \propto 4A^2 \cos^2\left(\frac{\pi d}{\lambda}\cdot\frac{x}{L}\right)$$

At $x = 0$, central maxima.

$x = \frac{\lambda L}{2d}(2n+1)$ for $n = 0, 1, 2, \ldots$, minima

* Adding $\Delta\phi$ introduces shift of the pattern.

Diffraction:



For an extended source of width D, we divide it into N segments with $d = D/N$.
$\Rightarrow$ There are $(N+1)$ beams (see figure); we are counting them from the edges. N is large.

Amplitude at a far away point (small angle holds and all angles are approximated by the average angle) is given by,

$$E = \frac{A}{(N+1)} e^{i(kr-\omega t)}\left[1 + e^{ikd\sin\theta} + e^{ik2d\sin\theta} + \cdots + e^{ikNd\sin\theta}\right]$$

$$= \frac{A}{(N+1)} e^{i(kr-\omega t)}\left(\frac{e^{ikNd\sin\theta}-1}{e^{ikd\sin\theta}-1}\right)$$

$$= \frac{A}{(N+1)} e^{i(kr-\omega t)} \frac{e^{i\frac{kNd}{2}\sin\theta}}{e^{i\frac{kd}{2}\sin\theta}} \cdot \frac{e^{i\frac{kNd}{2}\sin\theta} - e^{-i\frac{kNd}{2}\sin\theta}}{e^{i\frac{kd}{2}\sin\theta} - e^{-i\frac{kd}{2}\sin\theta}}$$

$$= \frac{A}{(N+1)} \cdot e^{i(kr-\omega t)} e^{ik(N-1)\frac{d}{2}\sin\theta} \cdot \frac{\sin\frac{Nkd\sin\theta}{2}}{\sin\frac{kd\sin\theta}{2}}$$

$$\Rightarrow I \propto |E|^2 = \frac{A^2}{(N+1)^2}\cdot\frac{\sin^2\frac{Nkd\sin\theta}{2}}{\sin^2\frac{kd\sin\theta}{2}} = \frac{A^2}{(N+1)^2}\cdot\frac{\sin^2 N\phi}{\sin^2\phi}$$

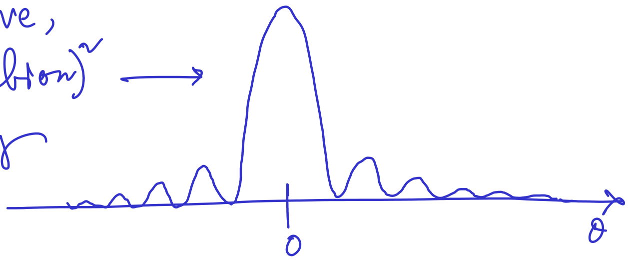
where, $\phi = \frac{kd\sin\theta}{2} = \pi\frac{d}{\lambda}\sin\theta$.

For very large N, we can have a small ϕ and $\sin\phi \sim \phi$ and $(N+1)^2 \sim N^2$

$$\Rightarrow I \propto \frac{A^2}{N^2} \cdot \frac{\sin^2\left(\pi \frac{Nd}{2\lambda} \sin\theta\right)}{\left(\pi \frac{d}{2\lambda} \sin\theta\right)^2} = A^2 \frac{\sin^2\left(\pi \frac{D}{2\lambda} \sin\theta\right)}{\left(\frac{\pi D}{2\lambda} \sin\theta\right)^2} \approx A^2 \frac{\sin^2\left(\frac{\pi D}{2\lambda}\theta\right)}{\left(\frac{\pi D}{2\lambda}\theta\right)^2}$$

For small angle

So, we have, a (sinc function)^2 like intensity distribution.



* Sinc function has a θ-dependent width

$\Rightarrow$ the beam "spreads" out as a result of the diffraction.